Asymptotic Notation, Review of Functions & Summations

Asymptotic Complexity

- Running time of an algorithm as a function of input size n **for large *n***.
- Expressed using only the **highest-order term** in the expression for the exact running time.
  - $7n^5 + 2n^4 + 3n^3 + 9n^2 + 4n + 6$
  - Instead of exact running time, we use *asymptotic notations* such as $O(n^5)$, $\Omega(n)$, $\Theta(n^2)$.
- Describes behavior of running time functions by setting lower and upper bounds for their values.

Asymptotic Notation

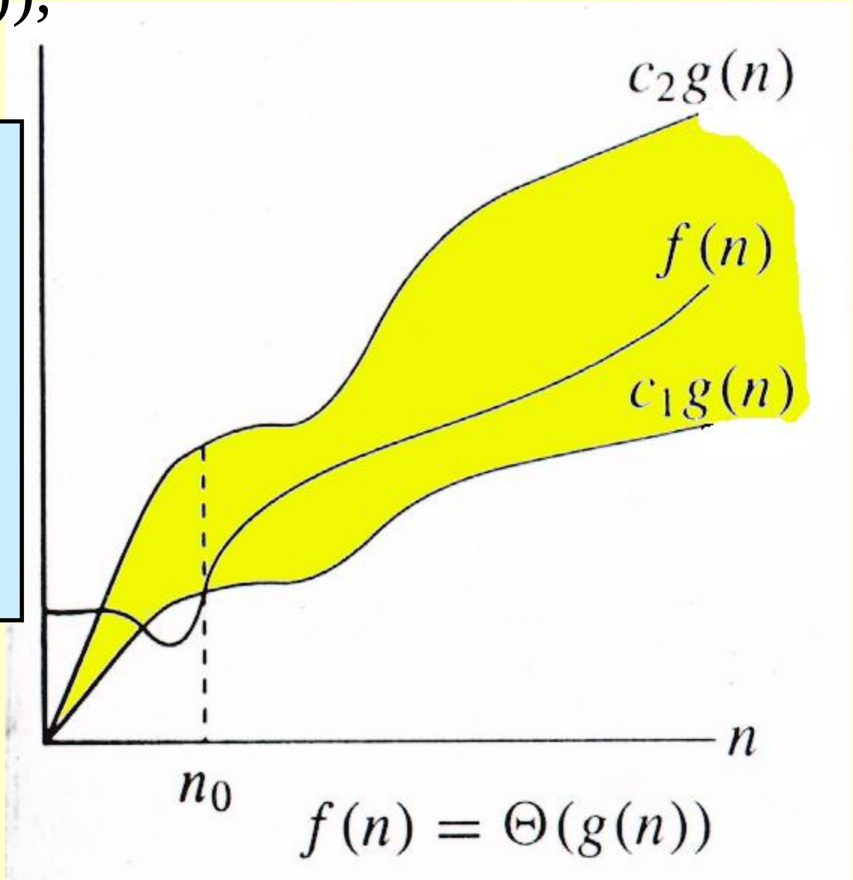
- **Θ, O, Ω, o, ω**
- Defined for functions over the natural numbers.
  - **Ex:** $f(n) = \Theta(n^2)$.
  - Describes how $f(n)$ grows in comparison to n^2.
- Define a ***set*** of functions; in practice used to compare two function values.
- The notations describe different rate-of-growth relations between the defining function and the defined set of functions.

Θ-notation

$g(n)$ = c (a constant), n, n^2, n^3, …

For function $g(n)$, we define $\Theta(g(n))$, big-Theta of n, as a set:

$\Theta(g(n)) = \{f(n)$:
$\exists$ positive constants c_1, c_2, and n_0,
such that $\forall n \geq n_0$,
we have $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)$
$\}$



Intuitively: Set of all functions that have the same *rate of growth* as $g(n)$.

$g(n)$ is an *asymptotically tight bound* for any $f(n)$ in the set.

Θ-notation

For function $g(n)$, we define $\Theta(g(n))$, big-Theta of n, as the set:

$\Theta(g(n)) = \{f(n) :$
$\exists$ positive constants c_1, c_2, and n_0,
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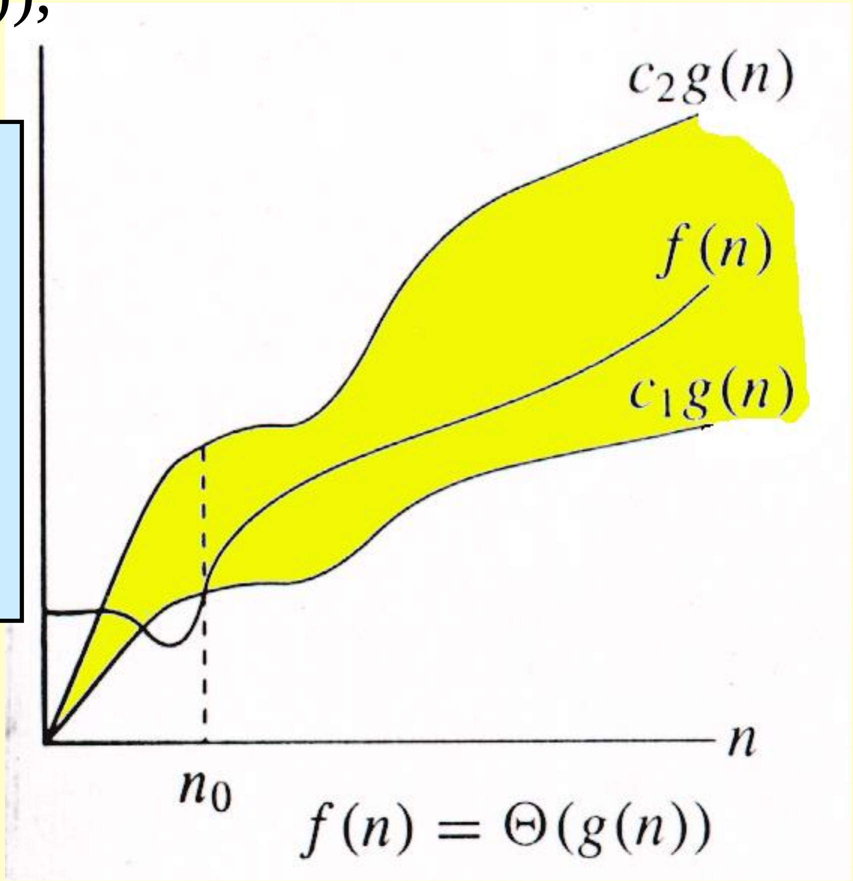


Technically, $f(n) \in \Theta(g(n))$.
Older usage, $f(n) = \Theta(g(n))$.
I'll accept either of the forms.

$f(n)$ and $g(n)$ are nonnegative, for large n.

Example

$\Theta(g(n)) = \{f(n)$: $\exists$ **positive constants** c_1, c_2, **and** n_0, **such that** $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- $10n^2 - 3n = \Theta(n^2)$?
- What constants for n_0, c_1, and c_2 will work?
- Make c_1 a little smaller than the leading coefficient, and c_2 a little bigger.
- ***To compare orders of growth, look at the leading term (highest-order term).***
- Exercise: Prove that $n^2/2\text{-}3n = \Theta(n^2)$

Example

$\Theta(g(n)) = \{f(n) :$ ∃ positive constants c_1, c_2, and n_0, such that $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- $10n^2 - 3n = \Theta(n^2)$?
- To show that this equation holds, we find $c_1 = 9$, $c_2 = 11$, and $n_0 = 3$ and for $n \geq n_0$, we always have

$$9n^2 \leq 10n^2 - 3n \leq 11n^2.$$

Example

$\Theta(g(n)) = \{f(n)$: $\exists$ positive constants c_1, c_2, and n_0, such that $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- $10n^2 - 3n = \Theta(n^2)$
- $10n^2 - 3n > 9n^2 \Rightarrow n^2 > 3n \Rightarrow n > 3$
- $10n^2 - 3n < 11n^2 \Rightarrow n^2 > -3n \Rightarrow n > -3$

Example

$\Theta(g(n)) = \{f(n) : \exists$ positive constants c_1, c_2, and n_0, such that $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- $n^2/2 - 3n = \Theta(n^2)$?
- $c_1 = 1/3 \Rightarrow n^2/2 - 3n > n^2/3$

  $\Rightarrow n^2/6 > 3n \quad \Rightarrow n > 18$
- $c_2 = 1 \quad \Rightarrow n^2/2 - 3n < n^2$

  $\Rightarrow n^2 > -6n \quad \Rightarrow n > -6$
- Then, for $n > n_0 = 18$, we will definitely have

  $n^2/3 < n^2/2 - 3n < n^2$.

Example

$\Theta(g(n)) = \{f(n) :$ $\exists$ positive constants c_1, c_2, and n_0, such that $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- Is $3n^3 \in \Theta(n^4)$?
- How about $2^{2n} \in \Theta(2^n)$?

Example

$\Theta(g(n)) = \{f(n) :$ $\exists$ positive constants c_1, c_2, and n_0, such that $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- Is $3n^3 \in \Theta(n^4)$?
- If it is true, we can find c_1, c_2, and n_0 such that for $n > n_0$, we have

  $$c_1 n^4 \leq 3n^3 \leq c_2 n^4.$$

  $$c_1 n^4 \leq 3n^3 \;\Rightarrow n \leq 3/c_1.$$

- It is a contradiction. So, $3n^3 \notin \Theta(n^4)$?

Example

$\Theta(g(n)) = \{f(n) :$ ∃ positive constants c_1, c_2, and n_0, such that $\forall n \geq n_0$, $0 \leq c_1 g(n) \leq f(n) \leq c_2 g(n)\}$

- How about $2^{2n} \in \Theta(2^n)$?
- If it is true, we can find c_1, c_2, and n_0 such that for $n > n_0$, we have

$$c_1 2^n \leq 2^{2n} \leq c_2 2^n.$$

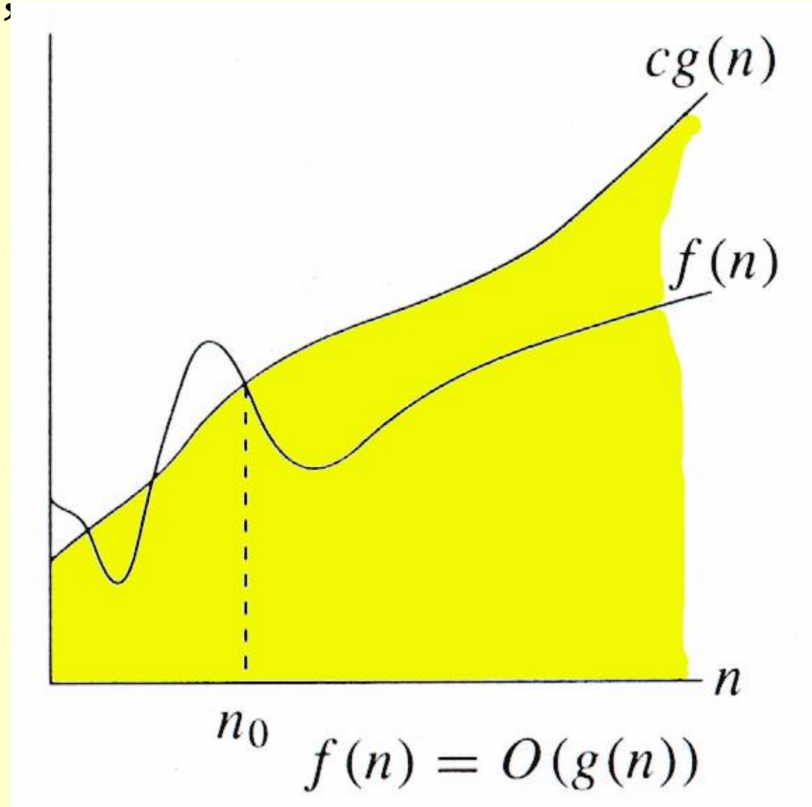
$$2^{2n} \leq c_2 2^n \quad \Rightarrow \quad 2^n \leq c_2 \quad \Rightarrow n \leq \log_2 c_2.$$

- It is a contradiction. So, $2^{2n} \notin \Theta(2^n)$?

O-notation

For function $g(n)$, we define $O(g(n))$, big-O of n, as the set:

> **$O(g(n)) = \{f(n) :$**
> **$\exists$ positive constants c and n_0,**
> **such that $\forall n \geq n_0$,**
> **we have $0 \leq f(n) \leq cg(n)$ $\}$**



Intuitively: Set of all functions whose *rate of growth* is the same as or lower than that of $g(n)$.

$g(n)$ is an *asymptotic upper bound* for any $f(n)$ in the set.

$f(n) = \Theta(g(n)) \Rightarrow f(n) = O(g(n))$.
$\Theta(g(n)) \subset O(g(n))$.

Examples

$O(g(n)) = \{f(n) :$ $\exists$ positive constants c and n_0, such that $\forall n \geq n_0$, we have $0 \leq f(n) \leq cg(n)$ $\}$

- Any linear *function* $an + b$ is in $O(n^2)$. **How?**
- Show that $3n^3 = O(n^4)$ for appropriate c and n_0.
- Show that $3n^3 = O(n^3)$ for appropriate c and n_0.

Examples

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$O(g(n)) = \{f(n) :$ ∃ positive constants c and n_0, such that $\forall n \geq n_0$, we have $0 \leq f(n) \leq cg(n)$ }

- Any linear *function* $an + b$ is in $O(n^2)$. **How?**
- To answer this question, we set $c = 1$, to see whether we have $an + b < n^2$ for $n >$ a constant n_0.
- To determine the value of n_0, we will solve an equation: $n^2 - an - b = 0$.
- We get $n_0 = \frac{a+\sqrt{a^2+4b}}{2}$

Examples

$O(g(n)) = \{f(n) :$ $\exists$ positive constants c and n_0, such that $\forall n \geq n_0$, we have $0 \leq f(n) \leq cg(n)$ $\}$

- Show that $3n^3 = O(n^4)$ for appropriate c and n_0.
- The answer is obviously *yes,* since for any $n > n_0 = 4$, we must have $n^4 > 3n^3$.

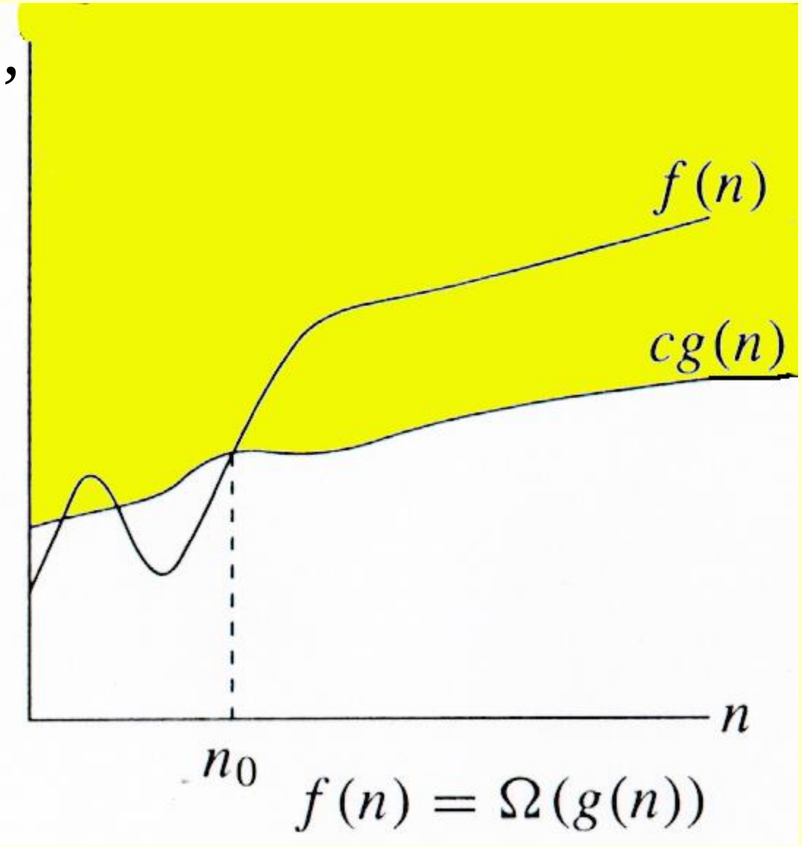
- Show that $3n^3 = O(n^3)$ for appropriate c and n_0.
- The answer is also *yes,* since we can take $c = 4$, and for any $n > n_0 = 1$, we must have $cn^3 > 3n^3$.

Ω -notation

For function $g(n)$, we define $\Omega(g(n))$, big-Omega of n, as the set:

$\Omega(g(n)) = \{f(n) :$
$\exists$ **positive constants** c **and** n_0,
such that $\forall n \geq n_0$,
we have $0 \leq cg(n) \leq f(n)\}$

Intuitively: Set of all functions whose *rate of growth* is the same as or higher than that of $g(n)$.



$g(n)$ is an *asymptotic lower bound* for any $f(n)$ in the set.

$f(n) = \Theta(g(n)) \Rightarrow f(n) = \Omega(g(n))$.
$\Theta(g(n)) \subset \Omega(g(n))$.

Example

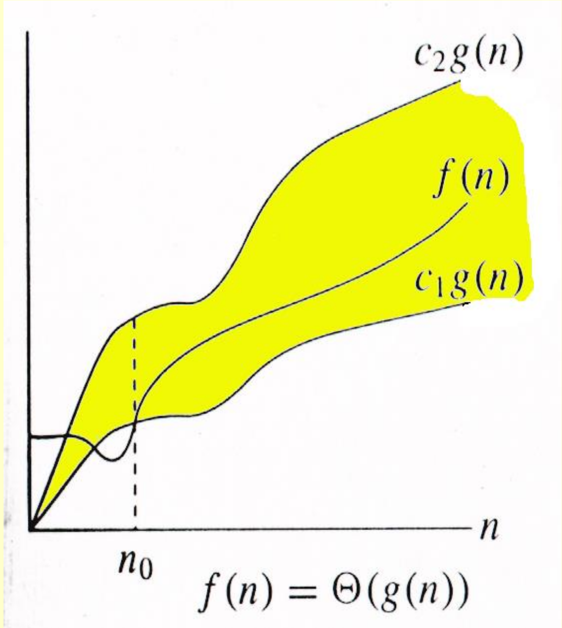
$\Omega(g(n)) = \{f(n) :$ $\exists$ positive constants c and n_0, such that $\forall n \geq n_0$, we have $0 \leq cg(n) \leq f(n)\}$

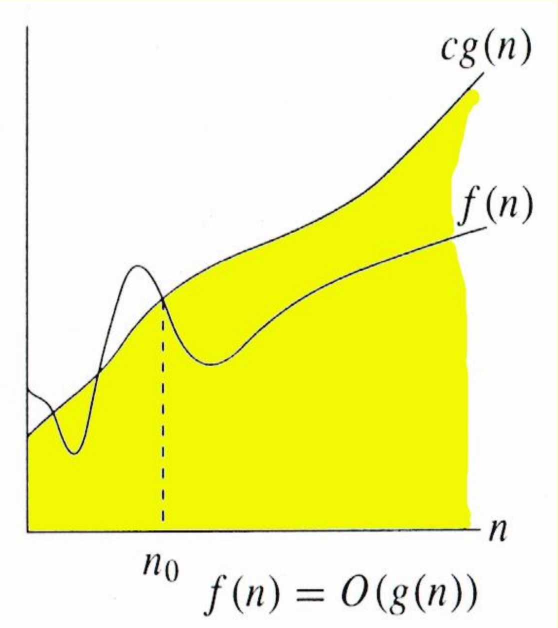
- $\sqrt{n} = \Omega(\log_2 n)$. Choose c and n_0.
- For this purpose, we need to determine constants c and n_0, such that for any $n \geq n_0$, we have

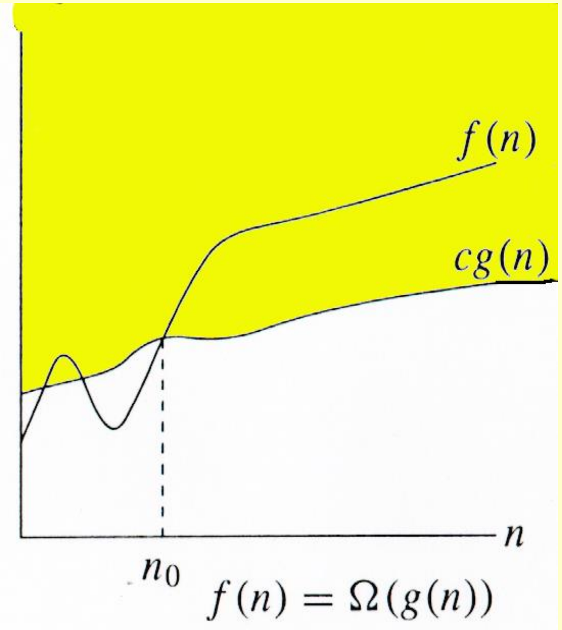
  $$C\log_2 n \leq \sqrt{n}$$

- We can $c = 1$ and $n_0 = 25$ since $\log_2 25 < \log_2 32 = 5 = \sqrt{25}$
- We can also prove that $\sqrt{n} - \log_2 n$ is an increasing function.

Relations Between Θ, *O*, Ω







Relations Between Θ, Ω, O

Theorem : For any two functions $g(n)$ and $f(n)$,

$f(n) = \Theta(g(n))$ iff

$f(n) = O(g(n))$ and $f(n) = \Omega(g(n))$.

- That is, $\Theta(g(n)) = O(g(n)) \cap \Omega(g(n))$
- In practice, asymptotically tight bounds are obtained from asymptotic upper and lower bounds.

Running Times

- "Running time is $O(f(n))$" $\Rightarrow$ Worst case is $O(f(n))$
- $O(f(n))$ bound on the worst-case running time $\Rightarrow$ $O(f(n))$ bound on the running time of every input.
- $\Theta(f(n))$ bound on the worst-case running time $\not\Rightarrow$ $\Theta(f(n))$ bound on the running time of every input.
- "Running time is $\Omega(f(n))$" $\Rightarrow$ Best case is $\Omega(f(n))$
- Can still say "Worst-case running time is $\Omega(f(n))$"
  - Means worst-case running time is given by some unspecified function $g(n) \in \Omega(f(n))$.

Example

- ***Insertion sort*** takes $\Theta(n^2)$ in the worst case, so sorting (as a *problem*) is $O(n^2)$. **Why?**
- Any sort algorithm must look at each item, so sorting is $\Omega(n)$.
- In fact, using (e.g.) merge sort, sorting is $\Theta(n \lg n)$ in the worst case.
  - Later, we will prove that we cannot hope that any comparison sort to do better in the worst case.

Asymptotic Notation in Equations

- Can use asymptotic notation in equations to replace expressions containing lower-order terms.
- For example,

  $4n^3 + 3n^2 + 2n + 1 = 4n^3 + 3n^2 + \Theta(n)$

  $= 4n^3 + \Theta(n^2) = \Theta(n^3)$. **How to interpret?**
- In equations, $\Theta(f(n))$ always stands for an ***anonymous function*** $g(n) \in \Theta(f(n))$
  - In the example above, $\Theta(n^2)$ stands for $3n^2 + 2n + 1$.

o-notation

For a given function $g(n)$, the set little-o:

$o(g(n)) = \{f(n): \forall c > 0, \exists n_0 > 0$ such that $\forall n \geq n_0$, we have $0 \leq f(n) < cg(n)\}$.

$f(n)$ becomes insignificant relative to $g(n)$ as n approaches infinity:

$$\lim_{n \to \infty} [f(n)/g(n)] = 0$$

$g(n)$ is an ***upper bound*** for $f(n)$ that is not asymptotically tight.

Observe the difference in this definition from previous ones. **Why?**

little-o:

$o(g(n)) = \{f(n)$:
$\forall c > 0, \exists n_0 > 0$ such that $\forall n \geq n_0$,
we have $0 \leq f(n) < cg(n)\}$.

big-O:

$O(g(n)) = \{f(n)$:
$\exists$ positive constants c and n_0, such that $\forall n \geq n_0$,
we have $0 \leq f(n) \leq cg(n)\ \}$

ω -notation

For a given function $g(n)$, the set little-omega:

$\omega(g(n)) = \{f(n)$: $\forall c > 0$, $\exists n_0 > 0$ such that $\forall n \geq n_0$, we have $0 \leq cg(n) < f(n)\}$.

$f(n)$ becomes arbitrarily large relative to $g(n)$ as n approaches infinity:

$$\lim_{n \to \infty} [f(n)/g(n)] = \infty.$$

$g(n)$ is a ***lower bound*** for $f(n)$ that is not asymptotically tight.

little-ω:

$\omega(\mathbf{g(n)}) = \{f(n)\text{: } \forall c > 0,\ \exists n_0 > 0 \text{ such that } \forall n \geq n_0,$
$\text{we have } 0 \leq cg(n) < f(n)\}.$

big-Ω:

$\Omega(g(n)) = \{f(n) :$
$\exists \text{ positive constants } c \text{ and } n_0, \text{ such that } \forall n \geq n_0,$
$\text{we have } 0 \leq cg(n) \leq f(n)\}$

Comparison of Functions

$$f \leftrightarrow g \;\approx\; a \leftrightarrow b$$

$$f(n) = O(g(n)) \;\approx\; a \leq b$$

$$f(n) = \Omega(g(n)) \;\approx\; a \geq b$$

$$f(n) = \Theta(g(n)) \;\approx\; a = b$$

$$f(n) = o(g(n)) \;\approx\; a < b$$

$$f(n) = \omega(g(n)) \;\approx\; a > b$$

Limits

- $\lim_{n \to \infty} [f(n) / g(n)] = 0 \Rightarrow f(n) \in o(g(n))$
- $\lim_{n \to \infty} [f(n) / g(n)] < \infty \Rightarrow f(n) \in O(g(n))$
- $0 < \lim_{n \to \infty} [f(n) / g(n)] < \infty \Rightarrow f(n) \in \Theta(g(n))$
- $0 < \lim_{n \to \infty} [f(n) / g(n)] \Rightarrow f(n) \in \Omega(g(n))$
- $\lim_{n \to \infty} [f(n) / g(n)] = \infty \Rightarrow f(n) \in \omega(g(n))$
- $\lim_{n \to \infty} [f(n) / g(n)]$ undefined $\Rightarrow$ can't say

Properties

- **Transitivity**

$$f(n) = \Theta(g(n)) \ \& \ g(n) = \Theta(h(n)) \Rightarrow f(n) = \Theta(h(n))$$
$$f(n) = O(g(n)) \ \& \ g(n) = O(h(n)) \Rightarrow f(n) = O(h(n))$$
$$f(n) = \Omega(g(n)) \ \& \ g(n) = \Omega(h(n)) \Rightarrow f(n) = \Omega(h(n))$$
$$f(n) = o\,(g(n)) \ \& \ g(n) = o\,(h(n)) \Rightarrow f(n) = o\,(h(n))$$
$$f(n) = \omega(g(n)) \ \& \ g(n) = \omega(h(n)) \Rightarrow f(n) = \omega(h(n))$$

- **Reflexivity**

$$f(n) = \Theta(f(n))$$
$$f(n) = O(f(n))$$
$$f(n) = \Omega(f(n))$$

Properties

- **Symmetry**

  $f(n) = \Theta(g(n))$ *iff* $g(n) = \Theta(f(n))$

- **Complementarity**

  $f(n) = O(g(n))$ *iff* $g(n) = \Omega(f(n))$

  $f(n) = o(g(n))$ *iff* $g(n) = \omega((f(n))$

Common Functions

Monotonicity

- $f(n)$ is
  - **monotonically increasing** if $m \leq n \Rightarrow f(m) \leq f(n)$.
  - **monotonically decreasing** if $m \geq n \Rightarrow f(m) \geq f(n)$.
  - **strictly increasing** if $m < n \Rightarrow f(m) < f(n)$.
  - **strictly decreasing** if $m > n \Rightarrow f(m) > f(n)$.

Exponentials

- **Useful Identities:**

$$a^{-1} = \frac{1}{a}$$

$$(a^m)^n = a^{mn}$$

$$a^m a^n = a^{m+n}$$

- **Exponentials and polynomials**

$$\lim_{n\to\infty} \frac{n^b}{a^n} = 0$$

$$\Rightarrow n^b = o(a^n)$$

Logarithms

$x = \log_b a$ is the exponent for $a = b^x$.

Natural log: $\ln a = \log_e a$

Binary log: $\lg a = \log_2 a$

$\lg^2 a = (\lg a)^2$

$\lg \lg a = \lg (\lg a)$

$$a = b^{\log_b a}$$

$$\log_c (ab) = \log_c a + \log_c b$$

$$\log_b a^n = n \log_b a$$

$$\log_b a = \frac{\log_c a}{\log_c b}$$

$$\log_b (1/a) = -\log_b a$$

$$\log_b a = \frac{1}{\log_a b}$$

$$a^{\log_b c} = c^{\log_b a}$$

Logarithms and exponentials – Bases

- If the base of a logarithm is changed from one constant to another, the value is altered by a constant factor.
  - **Ex:** $\log_{10} n * \mathbf{\log_2 10} = \log_2 n$.
  - Base of logarithm is not an issue in asymptotic notation.
- Exponentials with different bases differ by a exponential factor (not a constant factor).
  - **Ex:** $2^n = \mathbf{(2/3)^n} * 3^n$.

Polylogarithms

- **For $a \geq 0$, $b > 0$,** $\lim_{n\to\infty} (\lg^a n \,/\, n^b) = 0$,
  so $\lg^a n = o(n^b)$, and $n^b = \omega(\lg^a n)$
  - Prove using L'Hopital's rule repeatedly

- $\lg(n!) = \Theta(n \lg n)$
  - Prove using Stirling's approximation (in the text) for $\lg(n!)$.

Exercise

Express functions in A in asymptotic notation using functions in B.

A	B	
$5n^2 + 100n$	$3n^2 + 2$	$A \in \Theta(B)$

$A \in \Theta(n^2)$, $n^2 \in \Theta(B) \Rightarrow A \in \Theta(B)$

A	B	
$\log_3(n^2)$	$\log_2(n^3)$	$A \in \Theta(B)$

$\log_b a = \log_c a / \log_c b$; $A = 2\lg n / \lg 3$, $B = 3\lg n$, $A/B = 2/(3\lg 3)$

A	B	
$n^{\lg 4}$	$3^{\lg n}$	$A \in \omega(B)$

$a^{\log b} = b^{\log a}$; $B = 3^{\lg n} = n^{\lg 3}$; $A/B = n^{\lg(4/3)} \to \infty$ as $n \to \infty$

A	B	
$\lg^2 n$	$n^{1/2}$	$A \in o(B)$

$\lim_{n\to\infty} (\lg^a n / n^b) = 0$ (here $a = 2$ and $b = 1/2$) $\Rightarrow A \in o(B)$

Summations – Review

Review on Summations

- Why do we need summation formulas?

For computing the running times of iterative constructs (loops). (CLRS – Appendix A)

Example: Maximum Subvector

Given an array $A[1\ldots n]$ of numeric values (can be positive, zero, and negative) determine the subvector $A[i\ldots j]$ $(1 \le i \le j \le n)$ whose sum of elements is maximum over all subvectors.

1	-2	2	2

Review on Summations

```
MaxSubvector(A, n)
        maxsum ← 0;
        for i ← 1 to n
           do for j = i to n
                    sum ← 0
                    for k ← i to j
                        do sum += A[k]
                    maxsum ← max(sum, maxsum)
        return maxsum
```

- $T(n) = \sum_{i=1}^{n} \sum_{j=i}^{n} \sum_{k=i}^{j} 1$
- NOTE: This is not a simplified solution. What *is* the final answer?

Review on Summations

- **Constant Series:** For integers a and b, $a \le b$,

$$\sum_{i=a}^{b} 1 = b - a + 1$$

- **Linear Series (Arithmetic Series):** For $n \ge 0$,

$$\sum_{i=1}^{n} i = 1 + 2 + \cdots + n = \frac{n(n+1)}{2}$$

- **Quadratic Series:** For $n \ge 0$,

$$\sum_{i=1}^{n} i^2 = 1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$$

Review on Summations

- **Cubic Series:** For $n \geq 0$,

$$\sum_{i=1}^{n} i^3 = 1^3 + 2^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$$

- **Geometric Series:** For real $x \neq 1$,

$$\sum_{k=0}^{n} x^k = 1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}$$

For $|x| < 1$, $\quad \sum_{k=0}^{\infty} x^k = \frac{1}{1 - x}$

Review on Summations

- **Linear-Geometric Series:** For $n \ge 0$, real $c \neq 1$,

$$\sum_{i=1}^{n} ic^i = c + 2c^2 + \cdots + nc^n = \frac{-(n+1)c^{n+1} + nc^{n+2} + c}{(c-1)^2}$$

- **Harmonic Series:** nth harmonic number, $n \in \mathrm{I}^+$,

$$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

$$= \sum_{k=1}^{n} \frac{1}{k} = \ln(n) + O(1)$$

Review on Summations

- **Telescoping Series:**

$$\sum_{k=1}^{n} a_k - a_{k-1} = a_n - a_0$$

- **Differentiating Series:** For $|x| < 1$,

$$\sum_{k=0}^{\infty} kx^k = \frac{x}{(1-x)^2}$$

Review on Summations

- **Approximation by integrals:**
  - For monotonically increasing $f(n)$

$$\int_{m-1}^{n} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m}^{n+1} f(x)dx$$

  - For monotonically decreasing $f(n)$

$$\int_{m}^{n+1} f(x)dx \le \sum_{k=m}^{n} f(k) \le \int_{m-1}^{n} f(x)dx$$

- **How?**

Review on Summations

- ***n*th harmonic number**

$$\sum_{k=1}^{n} \frac{1}{k} \geq \int_{1}^{n+1} \frac{dx}{x} = \ln(n+1)$$

$$\sum_{k=2}^{n} \frac{1}{k} \leq \int_{1}^{n} \frac{dx}{x} = \ln n$$

$$\Rightarrow \sum_{k=1}^{n} \frac{1}{k} \leq \ln n + 1$$

Reading Assignment

- Chapter 4 of CLRS.