Heapsort

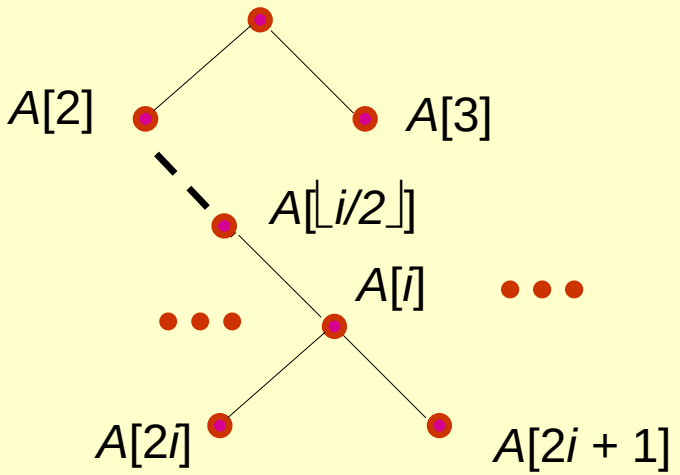
- What is a heap? Max-heap? Min-heap?
- Maintenance of Max-heaps
  - *MaxHeapify*
  - *BuildMaxHeap*
- Heapsort
  - Heapsort
  - Analysis
- Priority queues
  - Maintenance of priority queues

Heapsort

- Combines the better attributes of merge sort and quick sort.
  - » Like merge sort, but unlike quick sort, running time is $O(n \lg n)$.
  - » Like quick sort, but unlike merge sort, sorts in place.
- Introduces an algorithm design technique
  - » Create data structure (*heap*) to manage information during the execution of an algorithm.
- The *heap* has other applications beside sorting.
  - » Priority Queues

Data Structure Binary Heap

- Array viewed as a nearly complete binary tree.
  - Physically – linear array.
  - Logically – binary tree, filled on all levels (except lowest.)
- Map from array elements to tree nodes and vice versa
  - Root – $A[1]$, Left[Root] – $A[2]$, Right[Root] – $A[3]$
  - Left[i] – $A[2i]$
  - Right[i] – $A[2i+1]$
  - Parent[i] – $A[\lfloor i/2 \rfloor]$

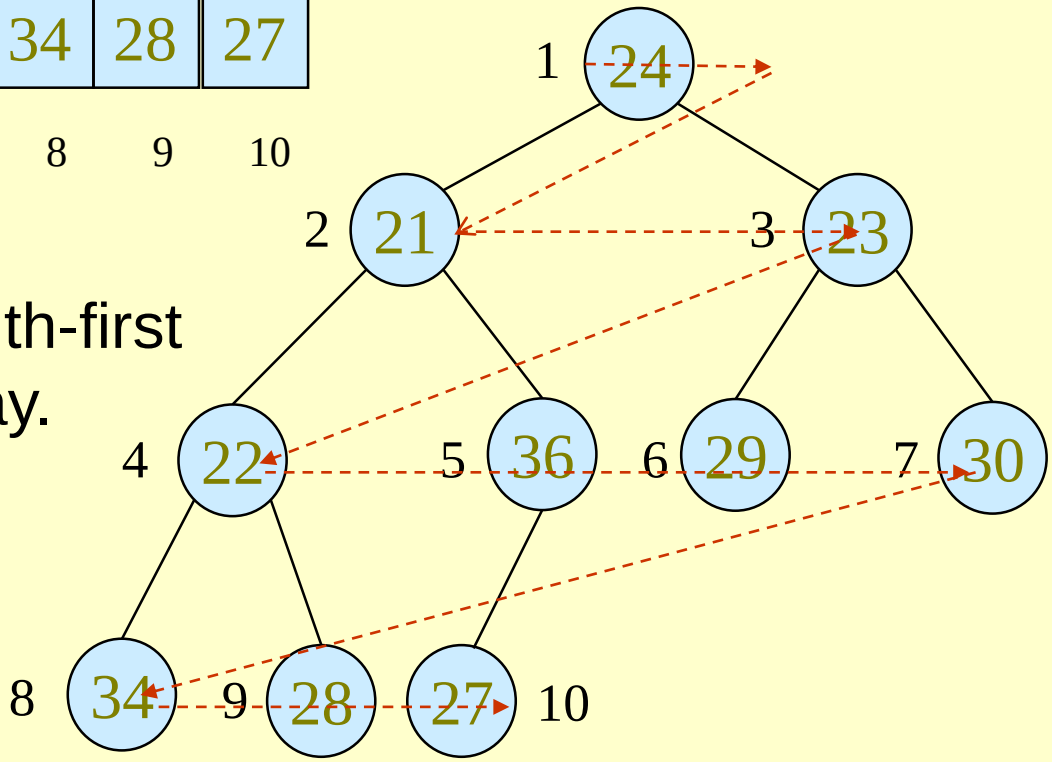


Data Structure Binary Heap

- length[*A*] – number of elements in array *A*.
- heap-size[*A*] – number of elements in heap stored in *A*.
  - » heap-size[*A*] ≤ length[*A*]

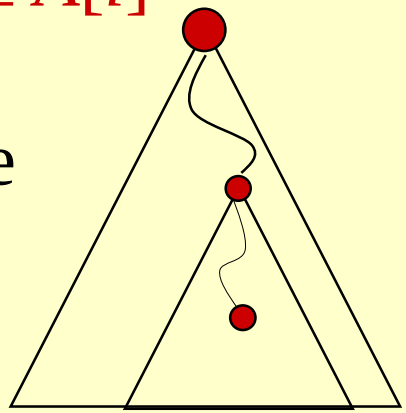
24	21	23	22	36	29	30	34	28	27
1	2	3	4	5	6	7	8	9	10

Searching the tree in breadth-first fashion, we will get the array.



Heap Property (Max and Min)

- Max-Heap
  - » For every node excluding the root, the value stored in that node is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree's root.



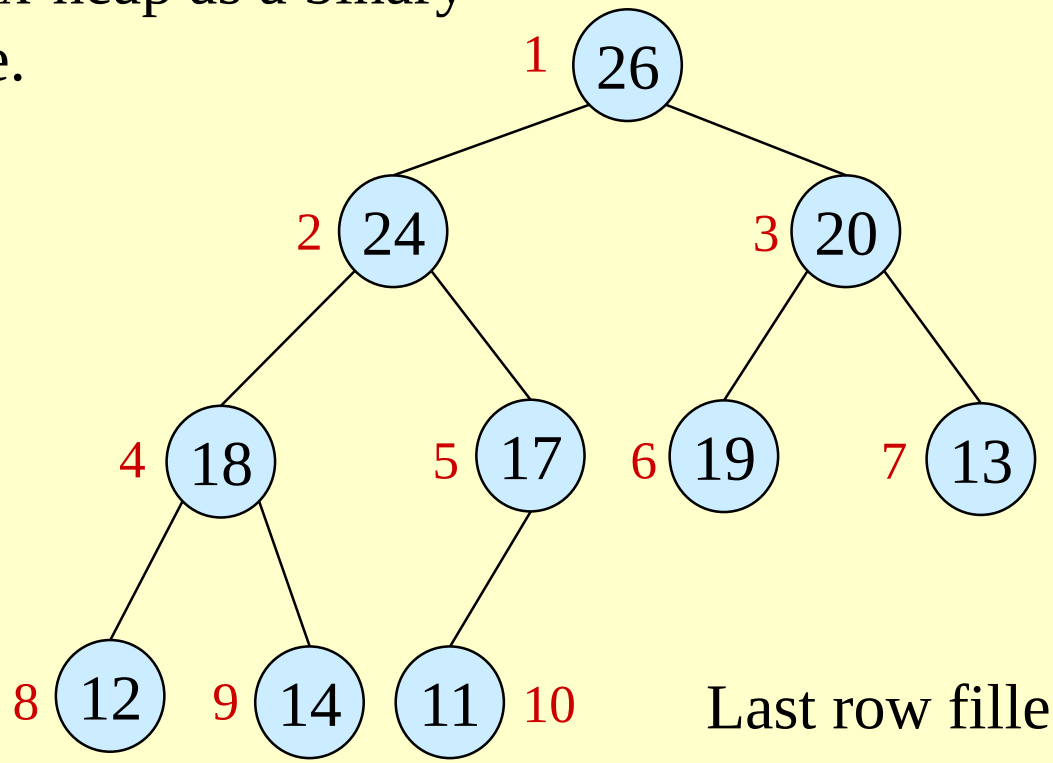
- Min-Heap
  - » For every node excluding the root, the value stored in that node is at least that of its parent: $A[parent[i]] \leq A[i]$
- Smallest element is stored at the root.
- In any subtree, no values are smaller than the value stored at subtree's root

Heaps – Example

26	24	20	18	17	19	13	12	14	11
1	2	3	4	5	6	7	8	9	10

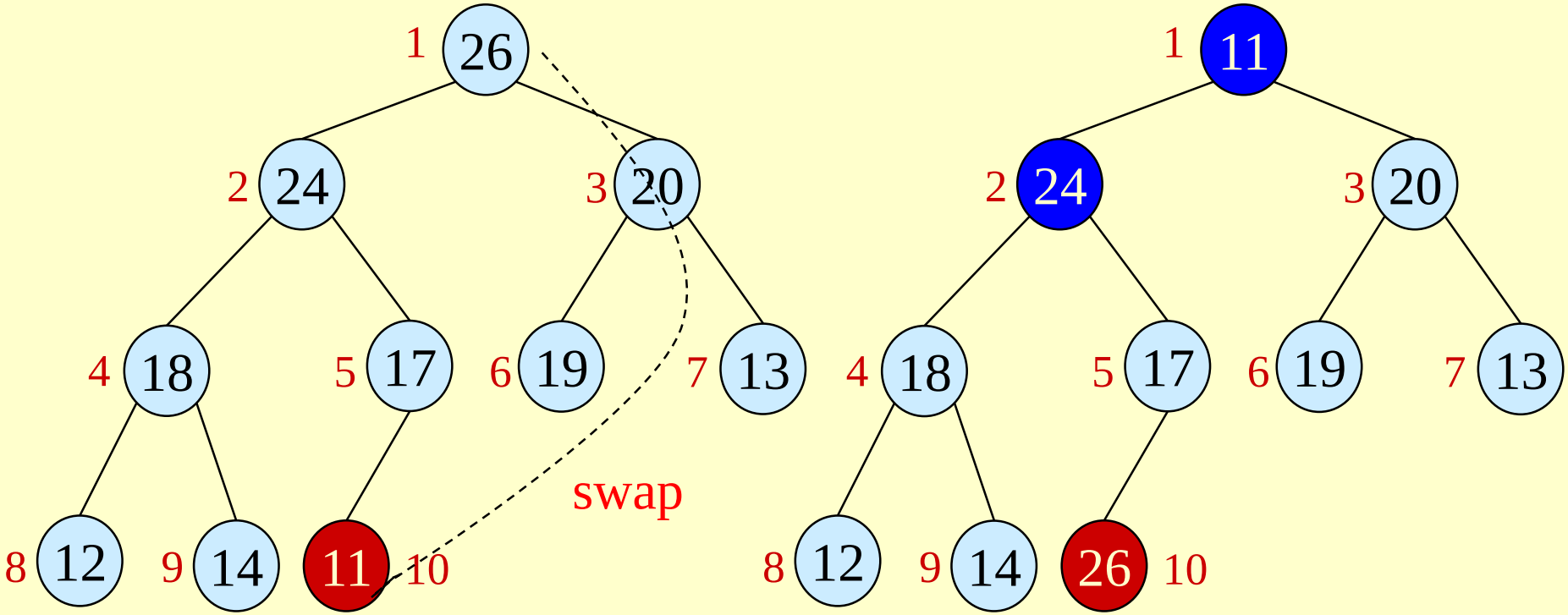
Max-heap as an array.

Max-heap as a binary tree.



Last row filled from left to right.

No longer a Max-heap:



Heaps in Sorting

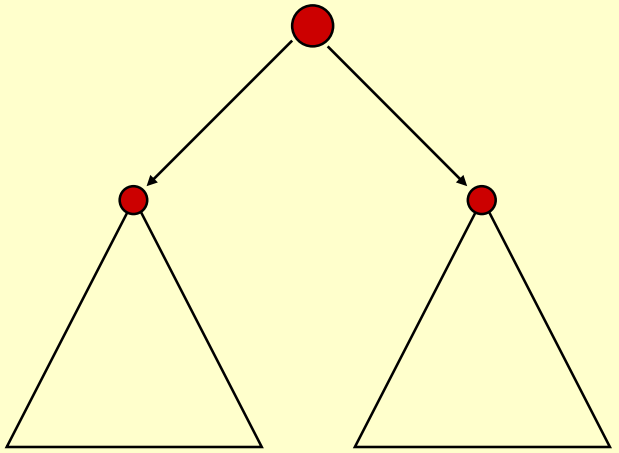
- Use max-heaps for sorting.
- The array representation of a max-heap is not sorted.
- Steps in sorting

  i) Convert a given array of size n to a max-heap (*BuildMaxHeap*)

  ii) Swap the first and last elements of the array.
    - Now, the largest element is in the last position – where it belongs.
    - That leaves $n - 1$ elements to be placed in their appropriate locations.
    - However, the array of first $n - 1$ elements is no longer a max-heap.
    - Float the element at the root down one of its subtrees so that the array remains a max-heap (*MaxHeapify*)
    - Repeat step (ii) until the array is sorted.

Maintaining the heap property

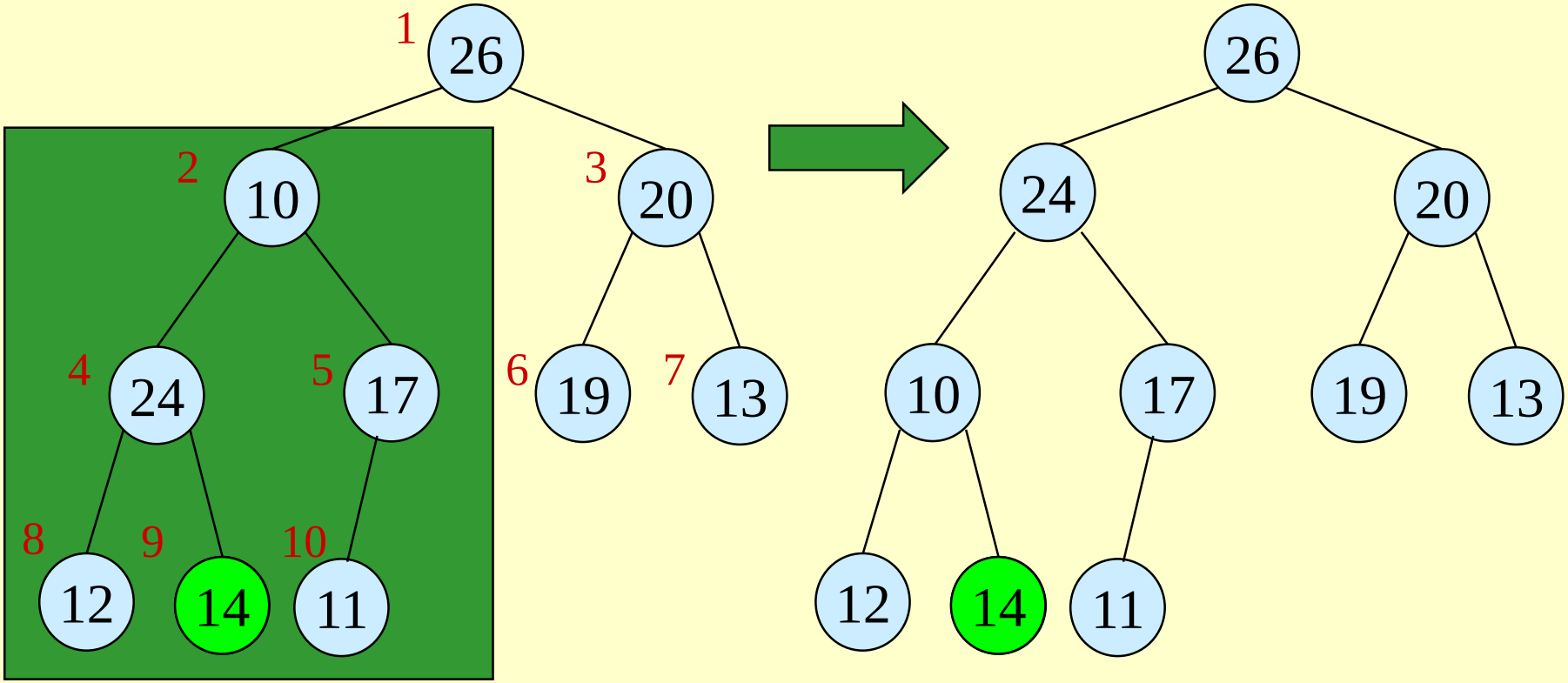
- Suppose two subtrees are max-heaps, but the root violates the max-heap property.

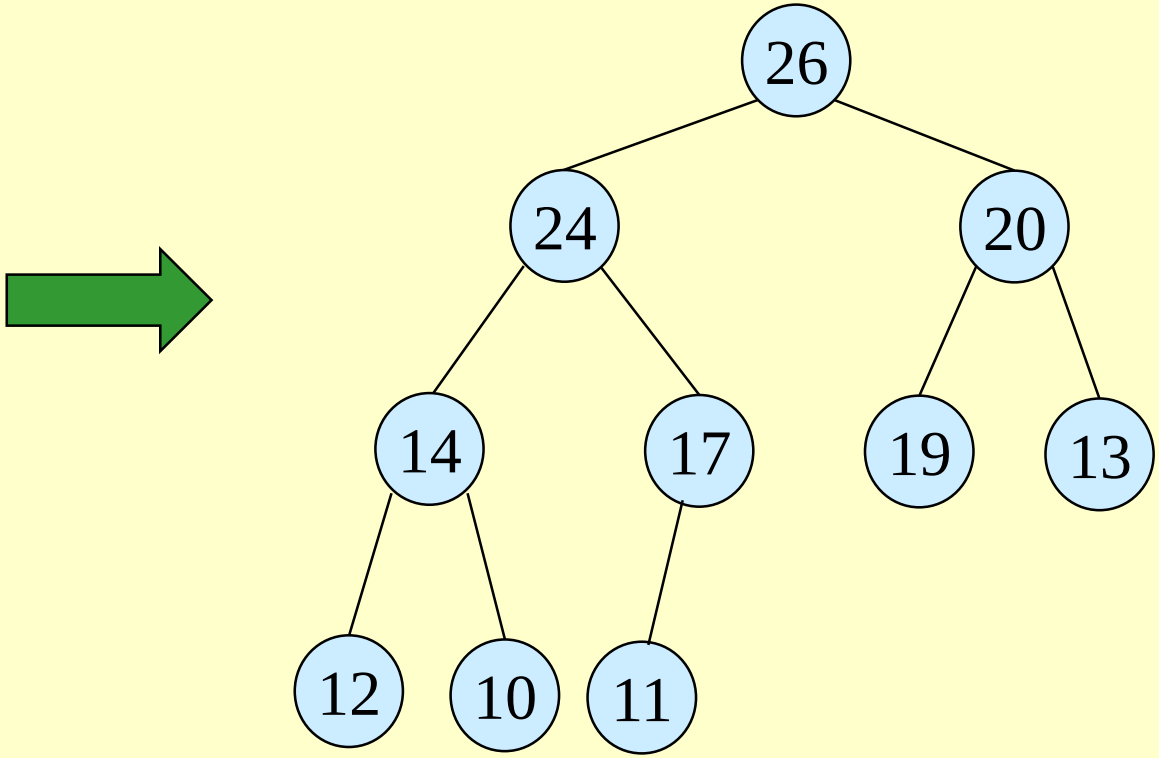


- Fix the offending node by exchanging the value at the node with the larger of the values at its children.
  - » May lead to the subtree at the child not being a max heap.
- Recursively fix the children until all of them satisfy the max-heap property.

MaxHeapify – Example

MaxHeapify(*A*, 2)

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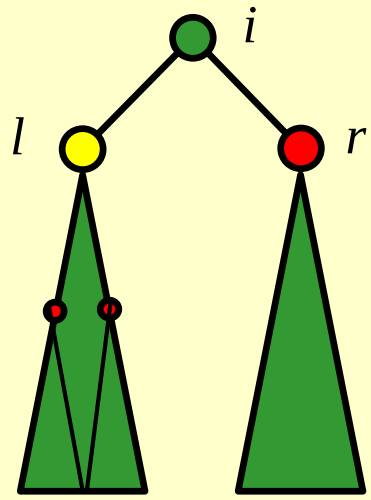
Procedure MaxHeapify

MaxHeapify(*A*, *i*)

1. $l \leftarrow$ left(i) (* $A[l]$ is the left child of $A[i]$.*)
2. $r \leftarrow$ right(i)
3. **if** $l \leq$ *heap-size*[A] and $A[l] > A[i]$
4. **then** *largest* $\leftarrow l$
5. **else** *largest* $\leftarrow i$
6. **if** $r \leq$ *heap-size*[A] **and** $A[r] > A[$*largest*$]$
7. **then** *largest* $\leftarrow r$
8. **if** *largest* $\neq i$
9. **then** exchange $A[i] \leftrightarrow A[$*largest*$]$
10. *MaxHeapify*(*A*, *largest*)

Assumption:
Left(i) and Right(i) are max-heaps.



A[*largest*] must be the largest among *A*[*i*], *A*[*l*] and *A*[*r*].

Running Time for MaxHeapify

MaxHeapify(*A*, *i*)

1. $l \leftarrow$ left(i)
2. $r \leftarrow$ right(i)
3. **if** $l \leq$ *heap-size*[A] and $A[l] > A[i]$
4. **then** *largest* $\leftarrow l$
5. **else** *largest* $\leftarrow i$
6. **if** $r \leq$ *heap-size*[A] **and** $A[r] > A[$*largest*$]$
7. **then** *largest* $\leftarrow r$
8. **if** *largest* $\neq i$
9. **then** exchange $A[i] \leftrightarrow A[$*largest*$]$
10. *MaxHeapify*(*A*, *largest*)

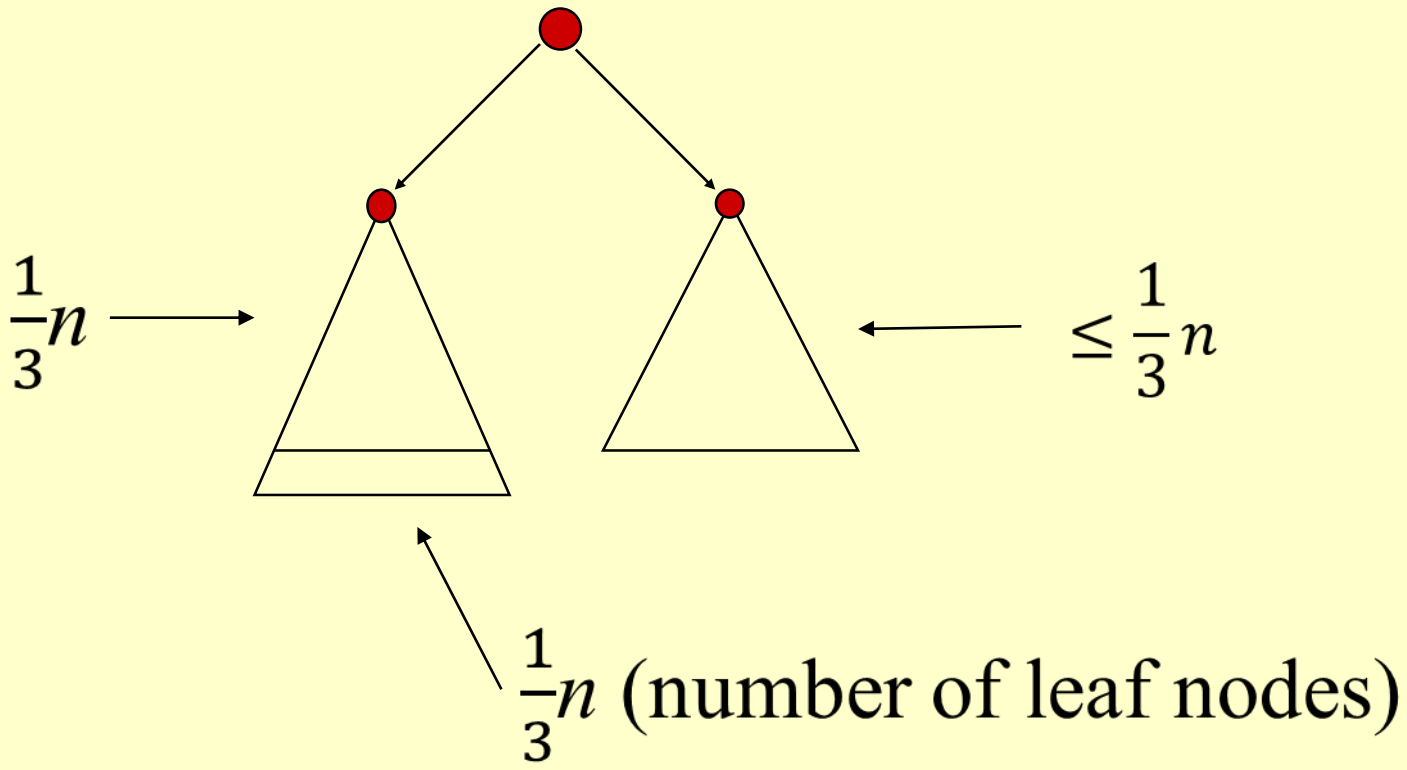
Time to fix node *i* and its children = **Θ(1)**

PLUS

Time to fix the subtree rooted at one of *i*'s children = *T*(size of subree at *largest*)

Running Time for MaxHeapify(A, n)

- $T(n) = T(\text{size of subree at } \textit{largest}) + \Theta(1)$
- size of subree at *largest* $\leq 2n/3$ (worst case occurs when the last row of tree is exactly half full)
- $T(n) \leq T(2n/3) + \Theta(1) \Rightarrow T(n) = O(\log_{1.5} n)$
- Alternately, MaxHeapify takes $O(h)$ where h is the height of the node where MaxHeapify is applied


$\frac{1}{3}n$
$\leq \frac{1}{3}n$
$\frac{1}{3}n$ (number of leaf nodes)

Building a heap

- Use *MaxHeapify* to convert an array *A* into a max-heap.
- How?
- Call MaxHeapify on each element in a bottom-up manner.

BuildMaxHeap(*A*)

1. *heap-size*[*A*] ← *length*[*A*]
2. **for** *i* ← $\lfloor$*length*[*A*]/2$\rfloor$ **downto** 1 (**A*[$\lfloor$*length*[*A*]/2$\rfloor$ +1],
3. **do** *MaxHeapify*(*A*, *i*) *A*[$\lfloor$*length*[*A*]/2$\rfloor$ +2],

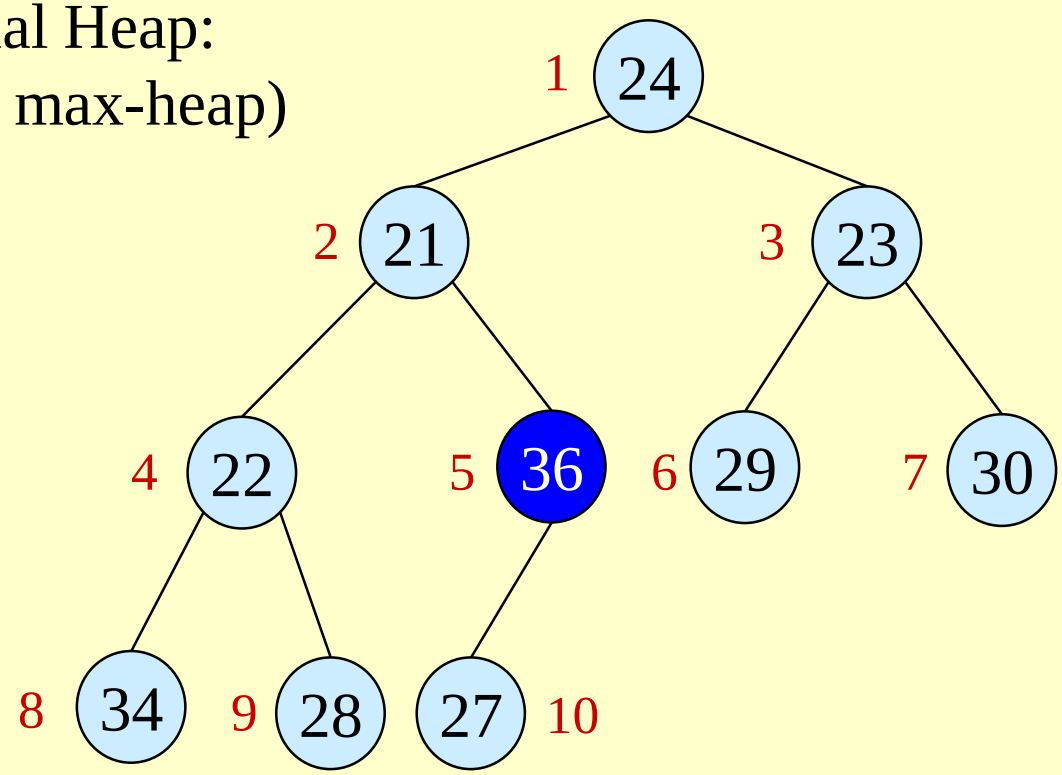
… are leaf nodes.*)

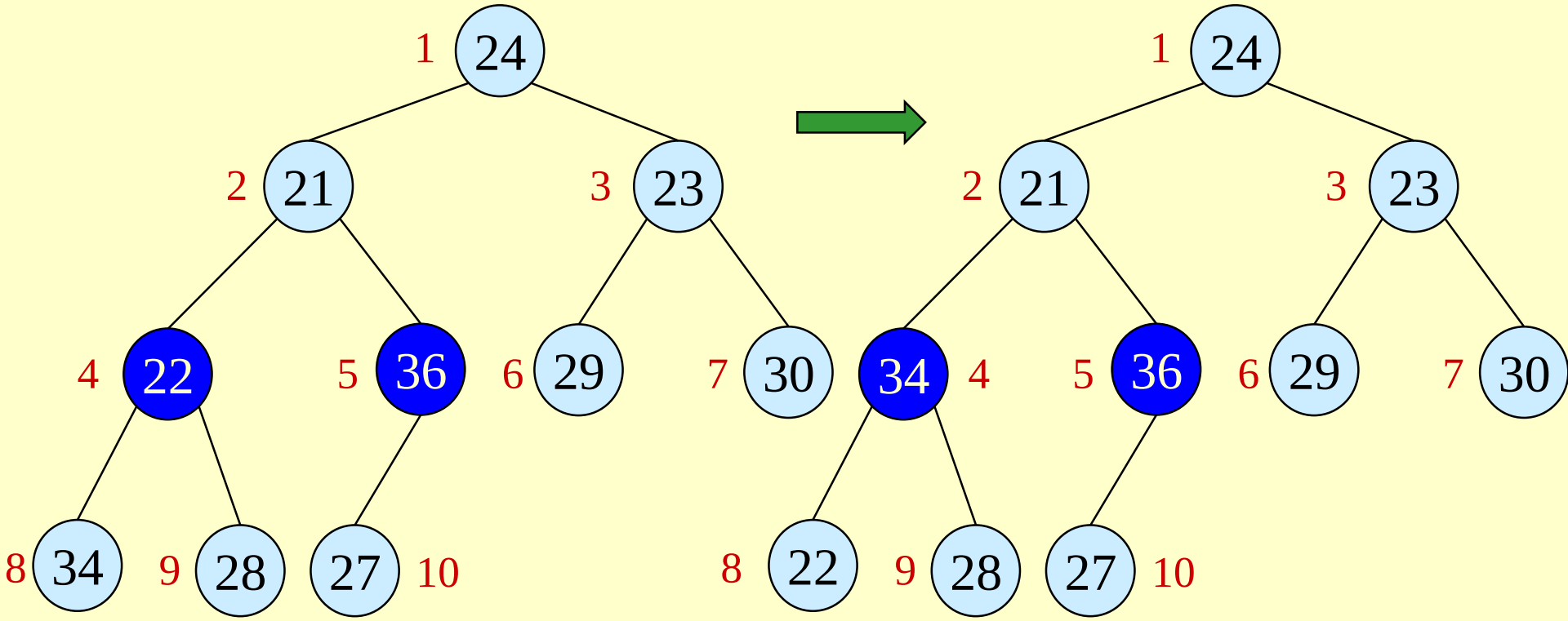
BuildMaxHeap – Example

Input Array:

24	21	23	22	36	29	30	34	28	27

Initial Heap:
(not max-heap)

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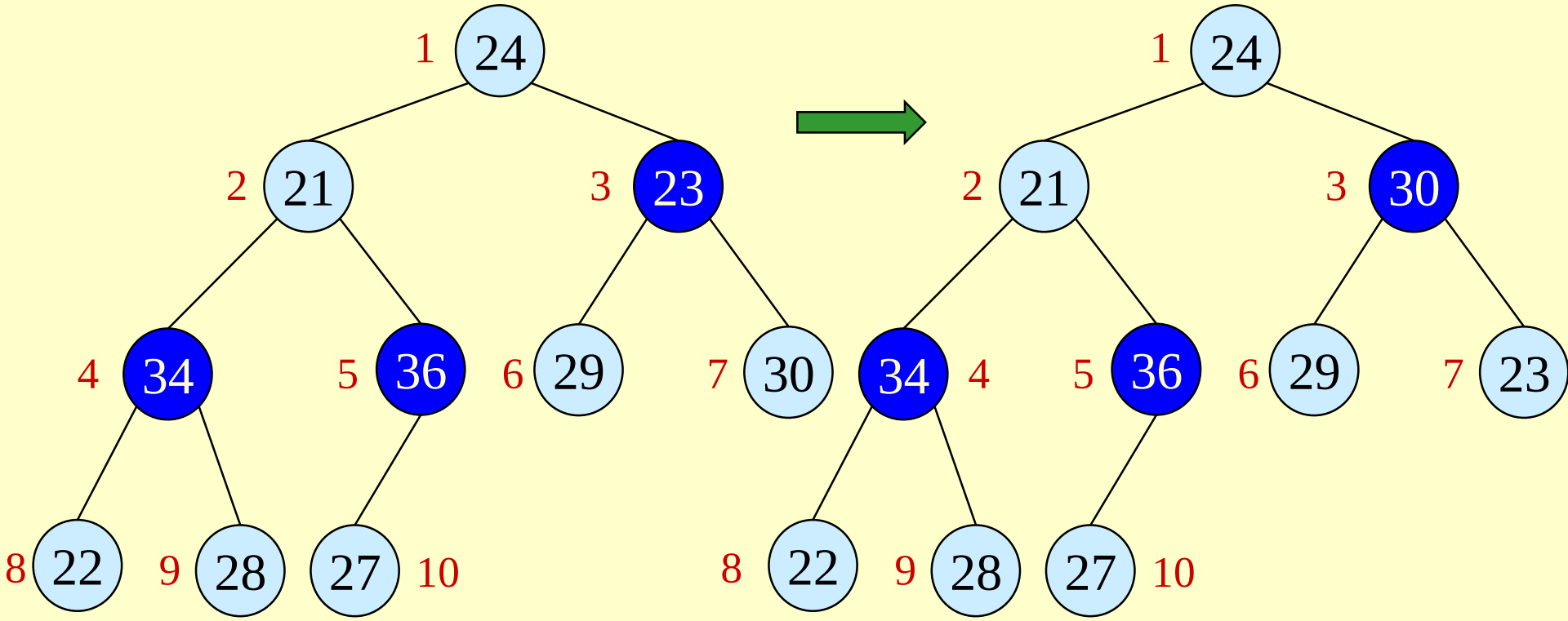
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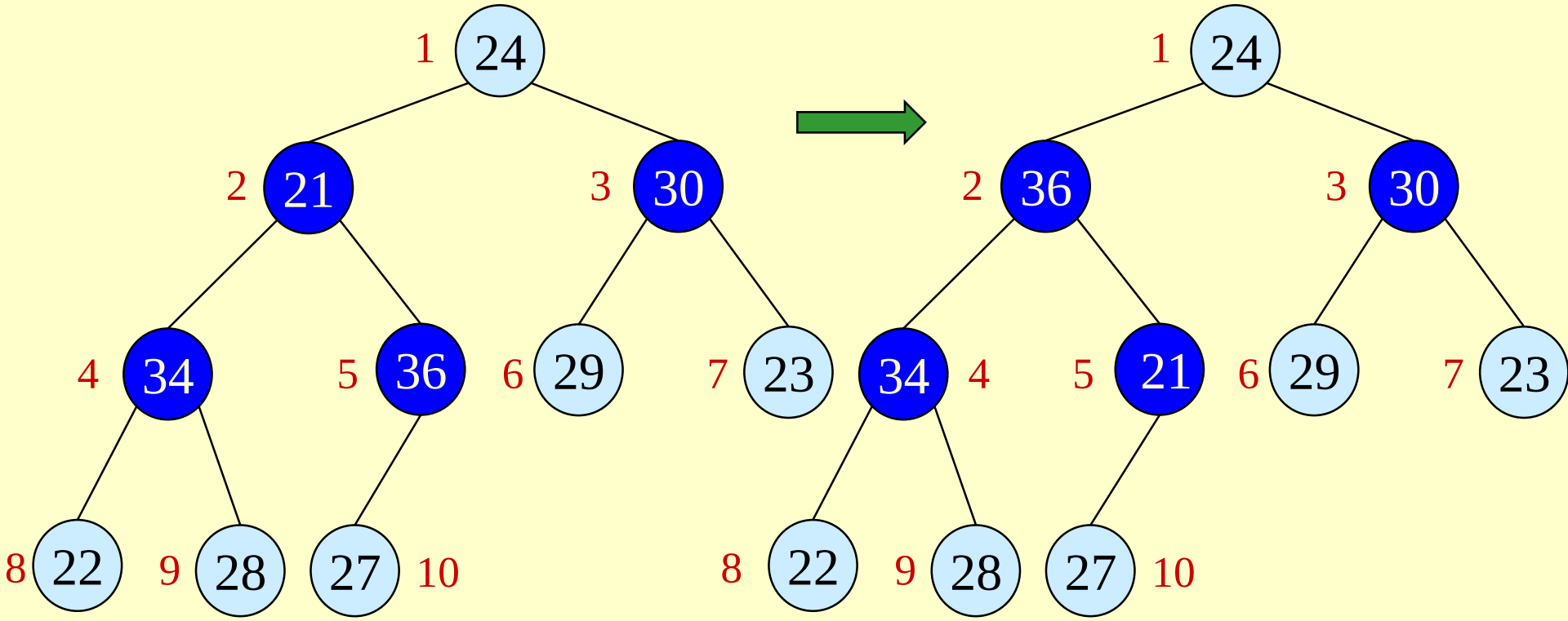
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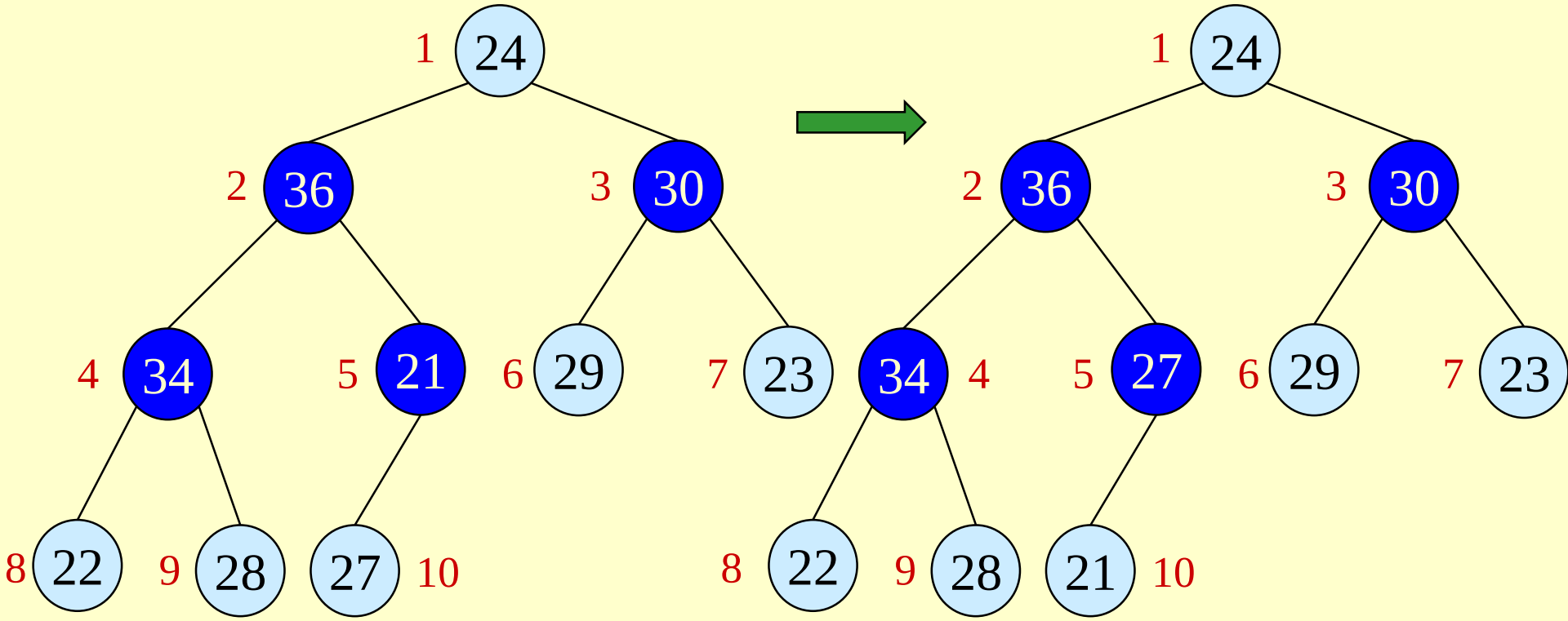
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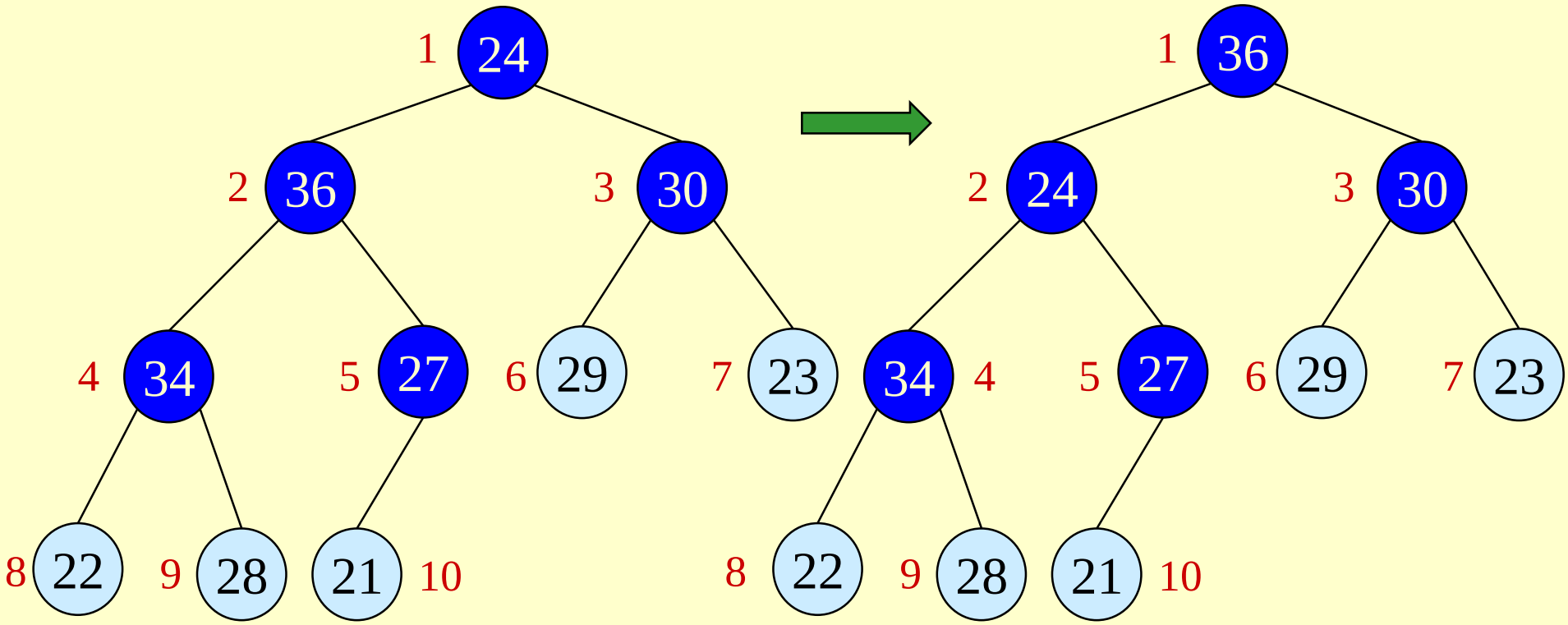
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BuildMaxHeap – Example

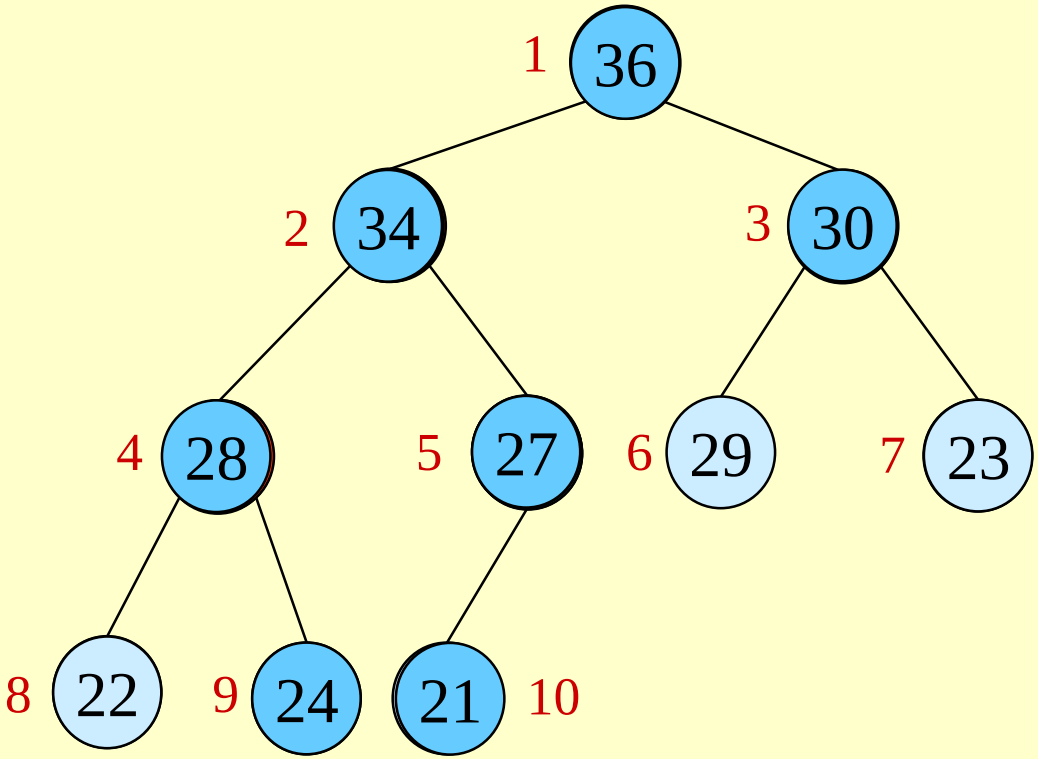
MaxHeapify(⌊10/2⌋ = 5)

MaxHeapify(4)

MaxHeapify(3)

MaxHeapify(2)

MaxHeapify(1)

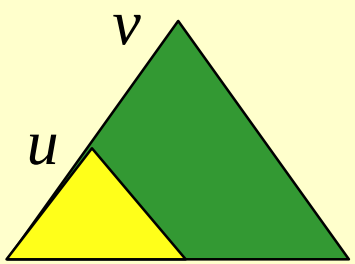


Correctness of *BuildMaxHeap*

- Loop Invariant: At the start of each iteration of the **for** loop, each node $i + 1$, $i + 2$, …, n is the root of a max-heap.
- Initialization:
  - » Before first iteration $i = \lfloor n/2 \rfloor$
  - » Nodes $\lfloor n/2 \rfloor+1, \lfloor n/2 \rfloor+2$, …, n are leaves and hence roots of max-heaps.
- Maintenance:
  - » By LI, subtrees at children of node i are max heaps.
  - » Hence, MaxHeapify(i) renders node i a max heap root (while preserving the max heap root property of higher-numbered nodes).
  - » Decrementing i reestablishes the loop invariant for the next iteration.
- Termination:
  - » On the termination, the root will be maxheapified. Thus, the whole tree is a Max-heap.

Running Time of *BuildMaxHeap*

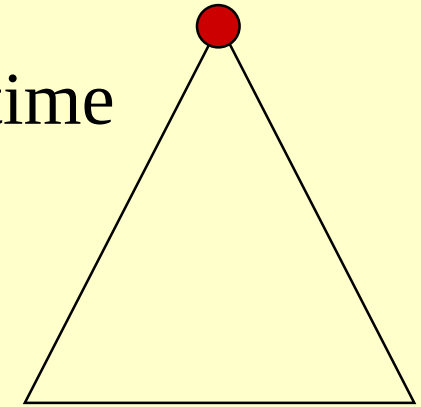
- Loose upper bound:
  - » Cost of a *MaxHeapify* call × No. of calls to *MaxHeapify*
  - » $O(\lg n) \times O(n) = O(n \lg n)$
- Tighter bound:
  - » Cost of a call to *MaxHeapify* at a node depends on the height, h, of the node – $O(h)$.
  - » Height of most nodes smaller than $\lfloor \lg n \rfloor$.
  - » Height of nodes h ranges from 0 to $\lfloor \lg n \rfloor$.
  - » No. of nodes of height h is at most $\lceil n/2^{h+1} \rceil$?



$$\text{Cost}_{maxheapify}(v) > \text{Cost}_{maxheapify}(u)$$

Height

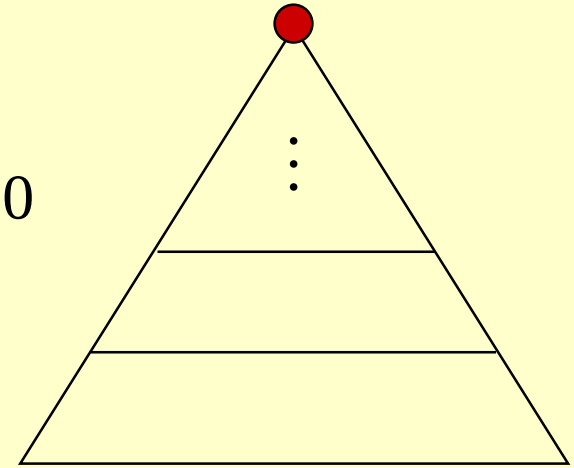
- *Height of a node in a tree*: the number of edges on the longest simple downward path from the node to a leaf.
- *Height of a tree*: the height of the root.
- Height of a heap: $\lfloor \lg n \rfloor$
  - » Basic operations on a heap run in $O(\lg n)$ time



Heap Characteristics

- *Height* $= h$
- No. of *leaves* $\leq \lceil n/2 \rceil$
- No. of nodes of height $h \leq \lceil n/2^{h+1} \rceil$?

height(*a leaf*) = 0



$\leq \lceil n/2^{2+1} \rceil$

$\leq \lceil n/2^{1+1} \rceil$

$\leq \lceil n/2^{0+1} \rceil$

Running Time of *BuildMaxHeap*

Tighter Bound for *T*(*BuildMaxHeap*)

T(*BuildMaxHeap*)

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \left\lceil \frac{n}{2^{h+1}} \right\rceil O(h)$$

$$= O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right)$$

$$\sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h}$$

$$\le \sum_{h=0}^{\infty} \frac{h}{2^h}$$

$$= \frac{1/2}{(1-1/2)^2}$$

$$= 2$$

$x = 1$ in (A.8)

$$O\left(n \sum_{h=0}^{\lfloor \lg n \rfloor} \frac{h}{2^h} \right) = O\left(n \sum_{h=0}^{\infty} \frac{h}{2^h} \right)$$

$$= O(n)$$

Can build a heap from an unordered array in linear time.

Heapsort

- Sort by maintaining the as yet unsorted elements as a max-heap.
- Start by building a max-heap on all elements in A.
  - » Maximum element is in the root, $A[1]$.
- Move the maximum element to its correct final position.
  - » Exchange $A[1]$ with $A[n]$.
- Discard $A[n]$ – it is now sorted.
  - » Decrement heap-size[A] by one.
- Restore the max-heap property on $A[1..n–1]$.
  - » Call *MaxHeapify*(A, 1).
- Repeat until heap-size[A] is reduced to 2.

Heapsort(*A*)

HeapSort(*A*)

1. Build-Max-Heap(*A*)
2. **for** $i \leftarrow$ *length*[*A*] **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. *heap-size*[*A*] $\leftarrow$ *heap-size*[*A*] – 1
5. *MaxHeapify*(*A*, 1)

Heapsort – Example

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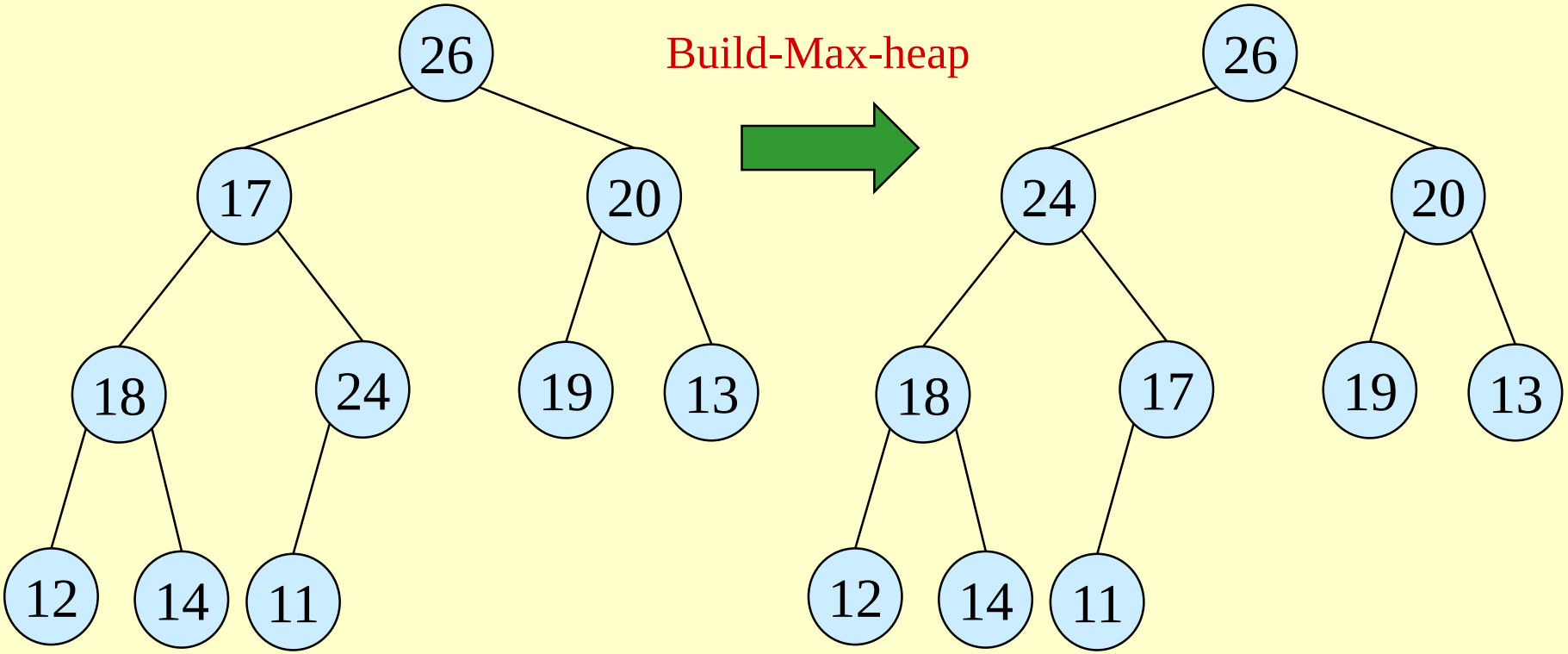

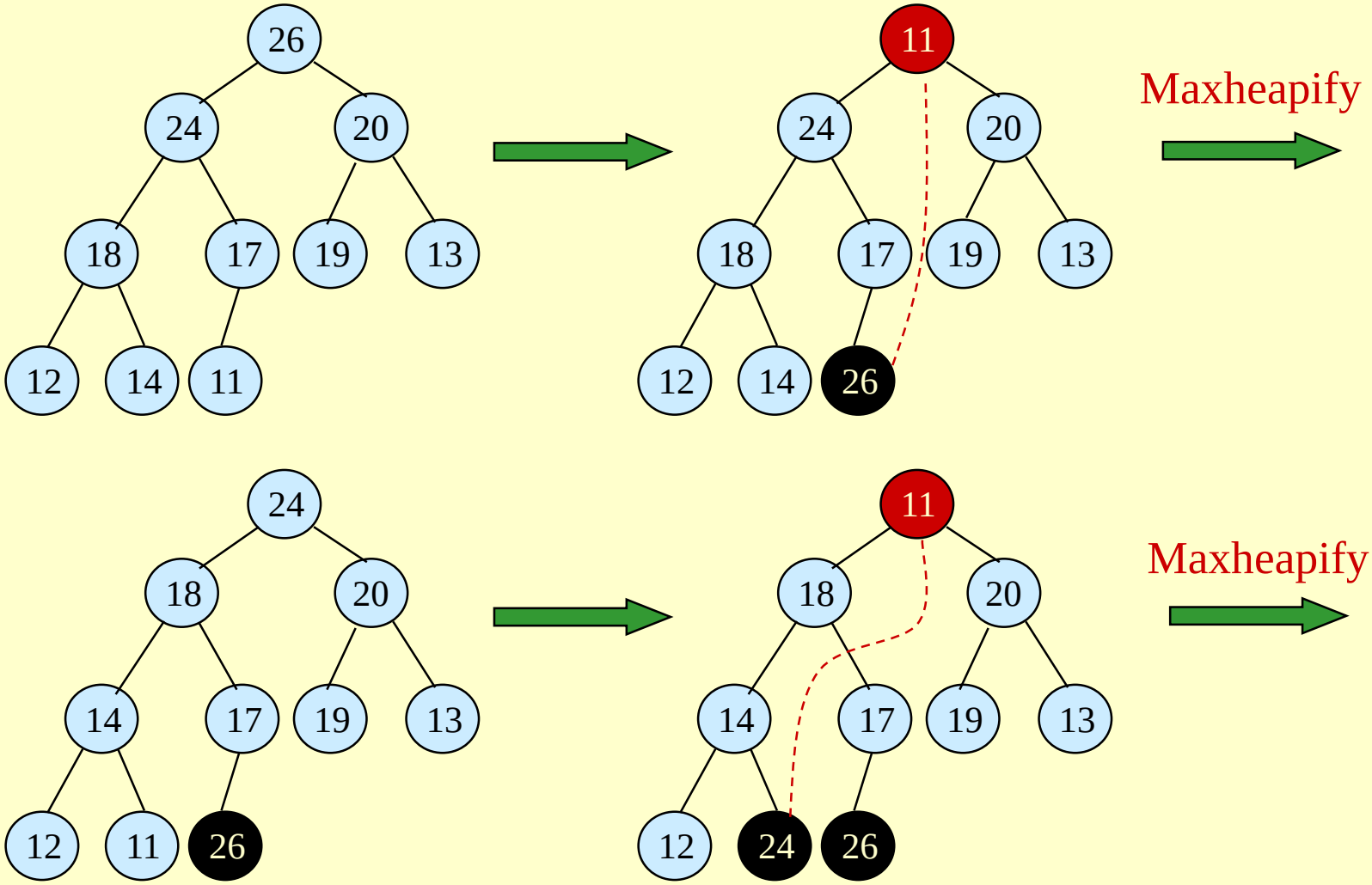
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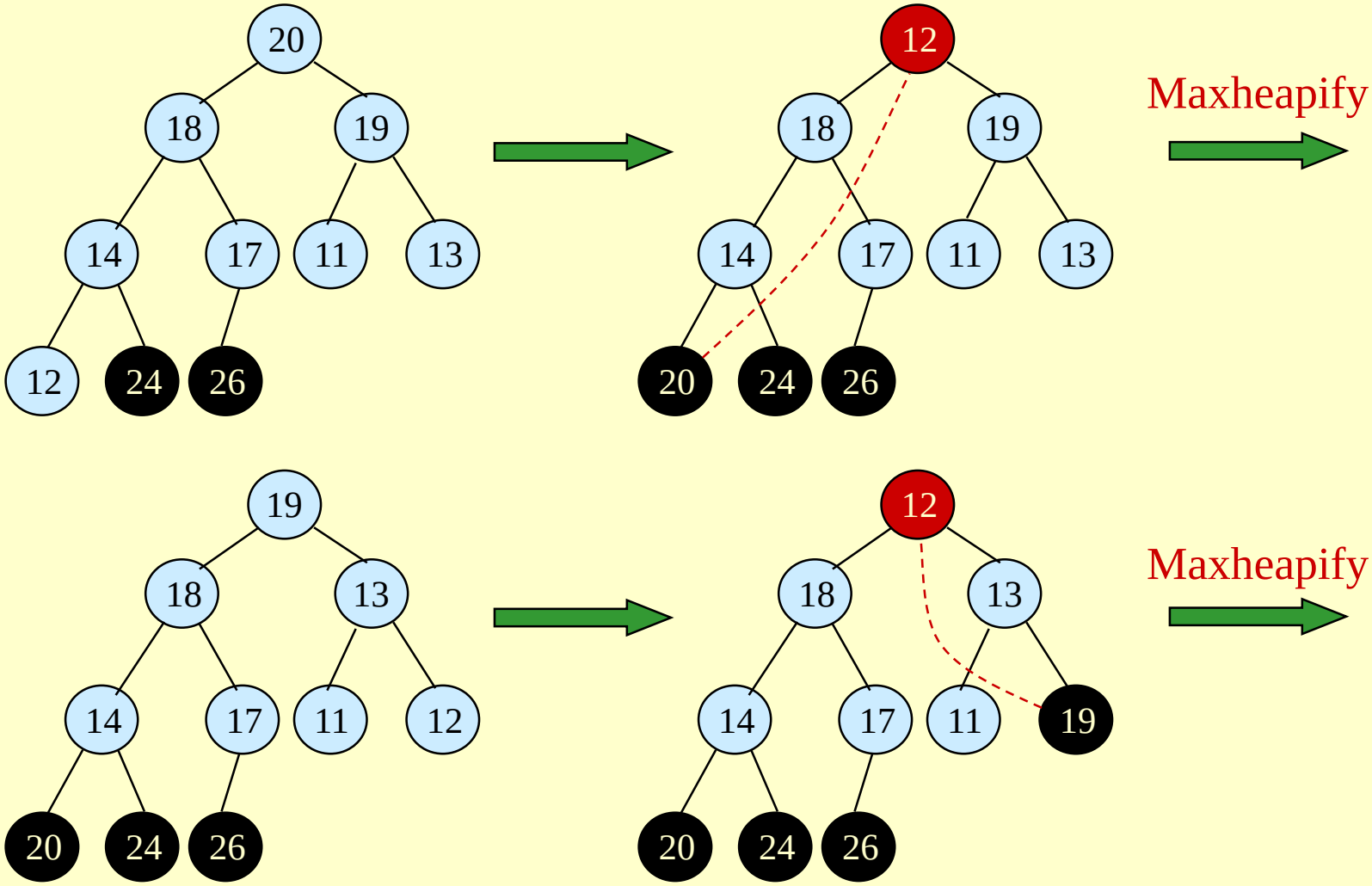
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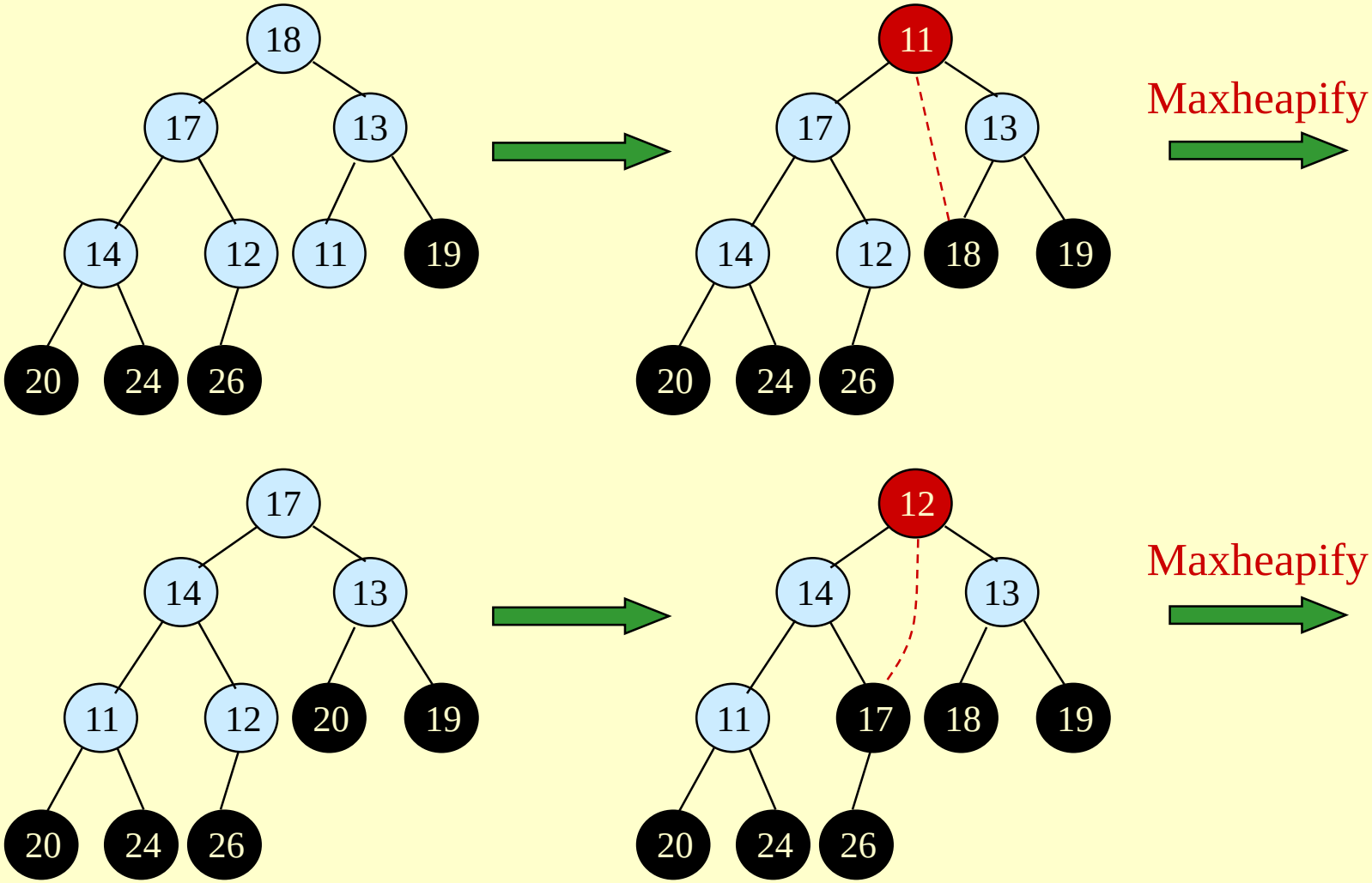
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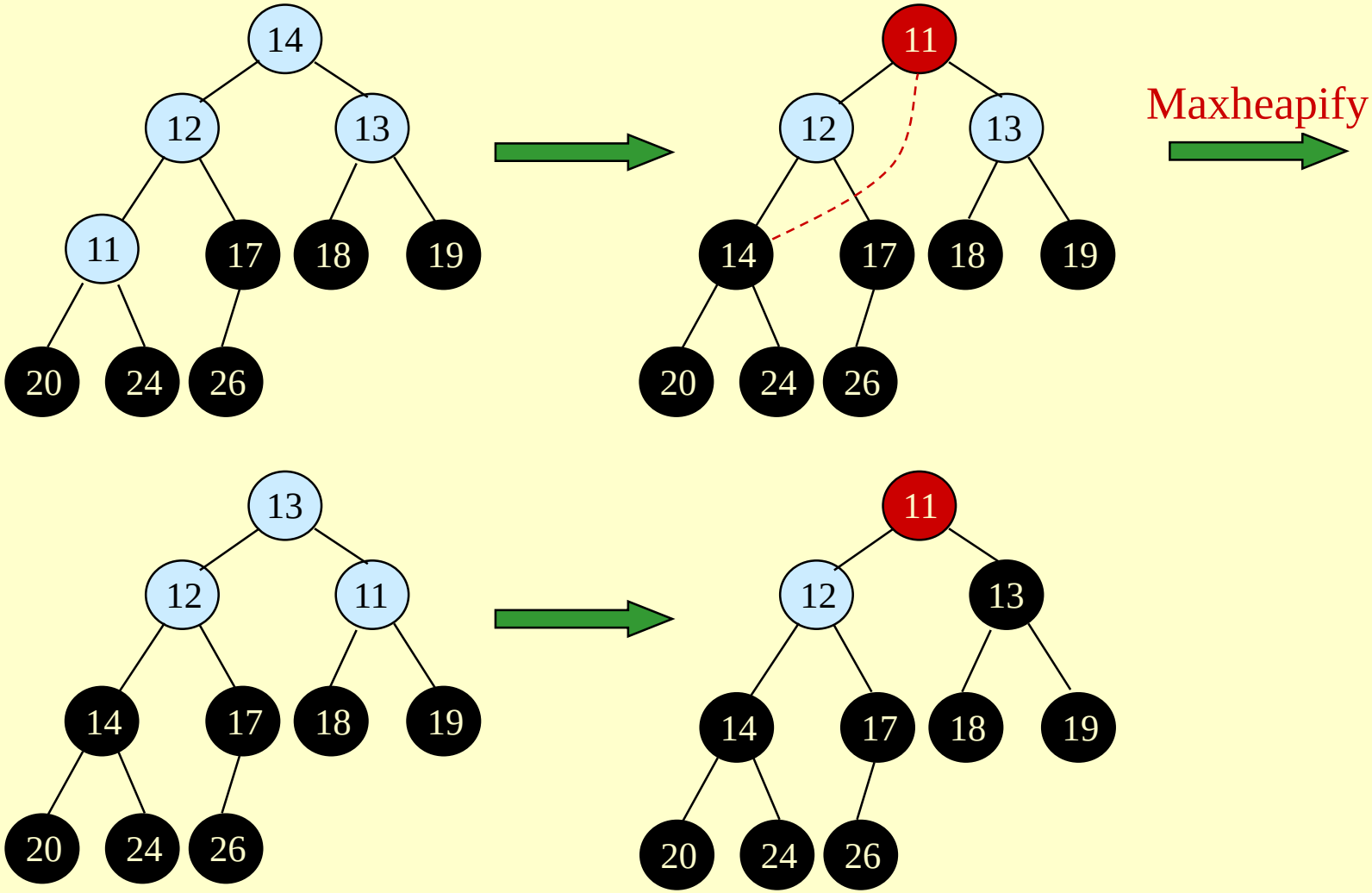
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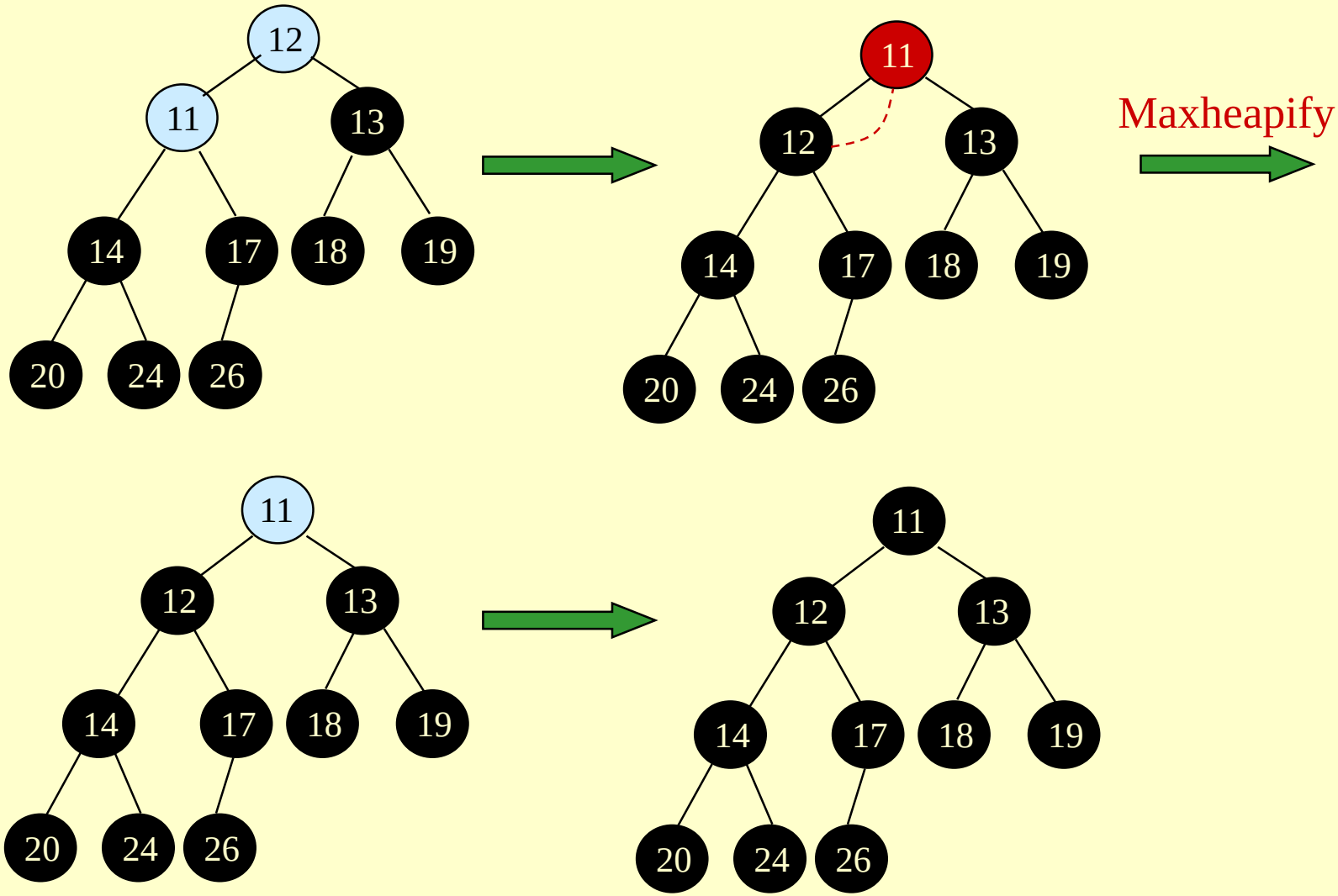
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Maxheapify
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Algorithm Analysis

HeapSort(*A*)

1. Build-Max-Heap(*A*)
2. **for** $i \leftarrow$ *length*[*A*] **downto** 2
3. **do** exchange $A[1] \leftrightarrow A[i]$
4. *heap-size*[*A*] $\leftarrow$ *heap-size*[*A*] – 1
5. *MaxHeapify*(*A*, 1)

- In-place
- Build-Max-Heap takes $O(n)$ and each of the n-1 calls to Max-Heapify takes time $O(\lg n)$.
- Therefore, $T(n) = O(n \lg n)$

Heap Procedures for Sorting

- MaxHeapify $O(\lg n)$
- BuildMaxHeap $O(n \lg n)$, $O(n)$?
- HeapSort $O(n \lg n)$

Priority Queue

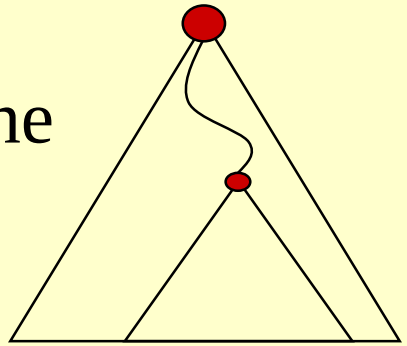
- Popular & important application of heaps.
- Max and min priority queues.
- Maintains a *dynamic* set *S* of elements.
- Each set element has a *key* – an associated value.
- Goal is to support insertion and extraction efficiently.
- Applications:
  - » Ready list of processes in operating systems by their priorities – the list is highly dynamic
  - » In event-driven simulators to maintain the list of events to be simulated in order of their time of occurrence.

Basic Operations

- Operations on a max-priority queue:
  - » Insert(S, x) - inserts the element x into the queue S
    - $S \leftarrow S \cup \{x\}$.
  - » Maximum(S) - returns the element of S with the largest key.
  - » Extract-Max(S) - removes and returns the element of S with the largest key.
  - » Increase-Key(S, x, k) – increases the value of element x's key to the new value k.
- Min-priority queue supports Insert, Minimum, Extract-Min, and Decrease-Key.
- Heap gives a good compromise between fast insertion but slow extraction and vice versa.

Heap Property (Max and Min)

- Max-Heap
  - » For every node excluding the root, the value stored in that node is at most that of its parent: $A[parent[i]] \geq A[i]$
- Largest element is stored at the root.
- In any subtree, no values are larger than the value stored at subtree root.



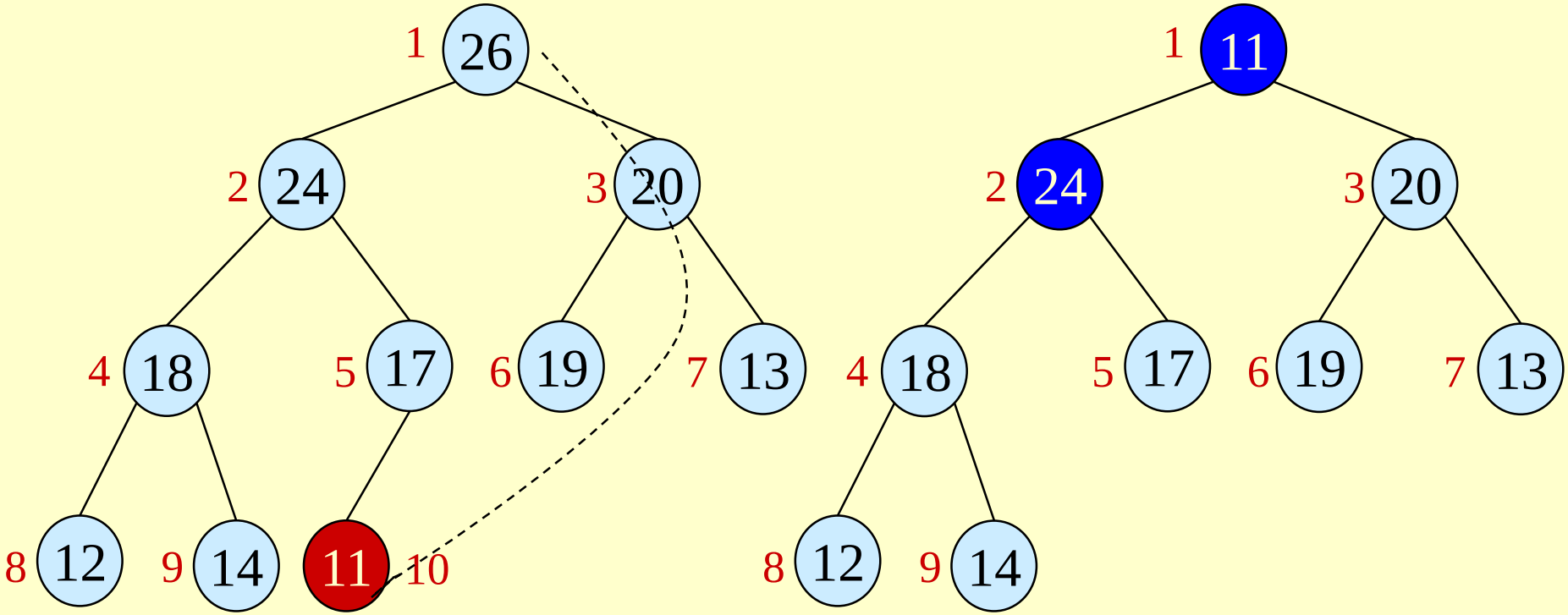
- Min-Heap
  - » For every node excluding the root, the value stored in that node is at least that of its parent: $A[parent[i]] \leq A[i]$
- Smallest element is stored at the root.
- In any subtree, no values are smaller than the value stored at subtree root

Heap-Extract-Max(A)

Implements the Extract-Max operation.

Heap-Extract-Max(A)
1. if *heap-size*[A] < 1
2. then error "heap underflow"
3. $max \leftarrow A[1]$
4. $A[1] \leftarrow A[$*heap-size*$[A]]$
5. *heap-size*$[A] \leftarrow$ *heap-size*$[A] - 1$
6. MaxHeapify(A, 1)
7. return *max*

Running time : Dominated by the running time of MaxHeapify
= $O(\lg n)$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Heap-Insert(*A*, *key*)

Heap-Insert(*A*, *key*)

1. *heap-size*[*A*] ← *heap-size*[*A*] + 1
2. *i* ← *heap-size*[*A*]
4. **while** *i* > 1 **and** *A*[Parent(*i*)] < *key*
5. **do** *A*[*i*] ← *A*[Parent(*i*)]
6. *i* ← Parent(*i*)
7. *A*[*i*] ← *key*

Running time is $O(\lg n)$

The path traced from the new leaf to the root has length $O(\lg n)$

Examples

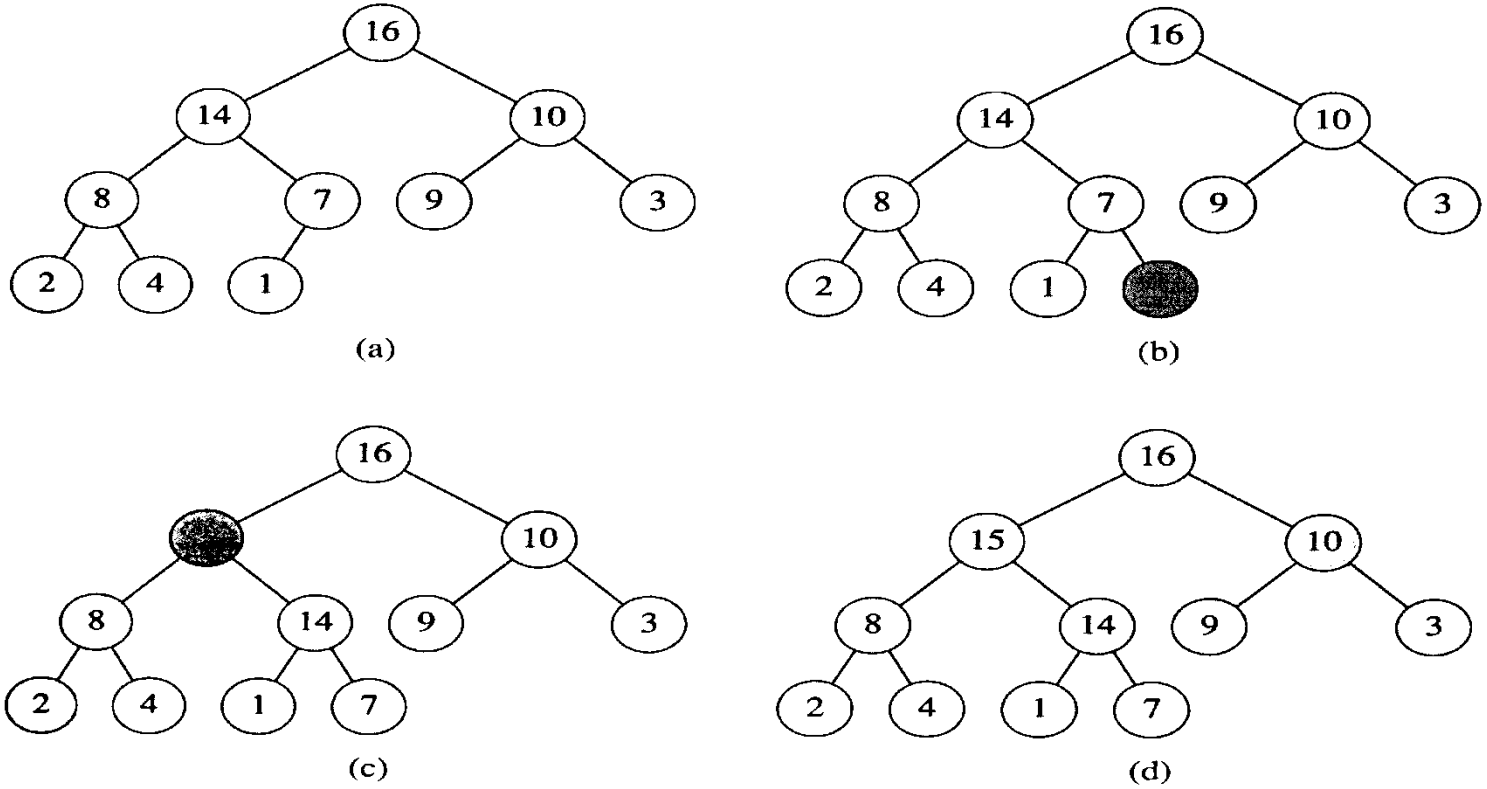

Figure 7.5 The operation of HEAP-INSERT. **(a)** The heap of Figure 7.4(a) before we insert a node with key 15. **(b)** A new leaf is added to the tree. **(c)** Values on the path from the new leaf to the root are copied down until a place for the key 15 is found. **(d)** The key 15 is inserted.

Heap-Increase-Key(*A*, *i*, *key*)

Heap-Increase-Key(*A*, *i*, *key*)

```
1    If key < A[i]
2       then error "new key is smaller than the current key"
3    A[i] ← key
4    while i > 1 and  A[Parent[i]] < A[i]
5          do exchange A[i] ↔ A[Parent[i]]
6             i ← Parent[i]
```

Heap-Insert(*A*, *key*)

```
1    heap-size[A] ← heap-size[A] + 1
2    A[heap-size[A]] ← – ∞
3    Heap-Increase-Key(A, heap-size[A], key)
```