Graph Algorithms – 2

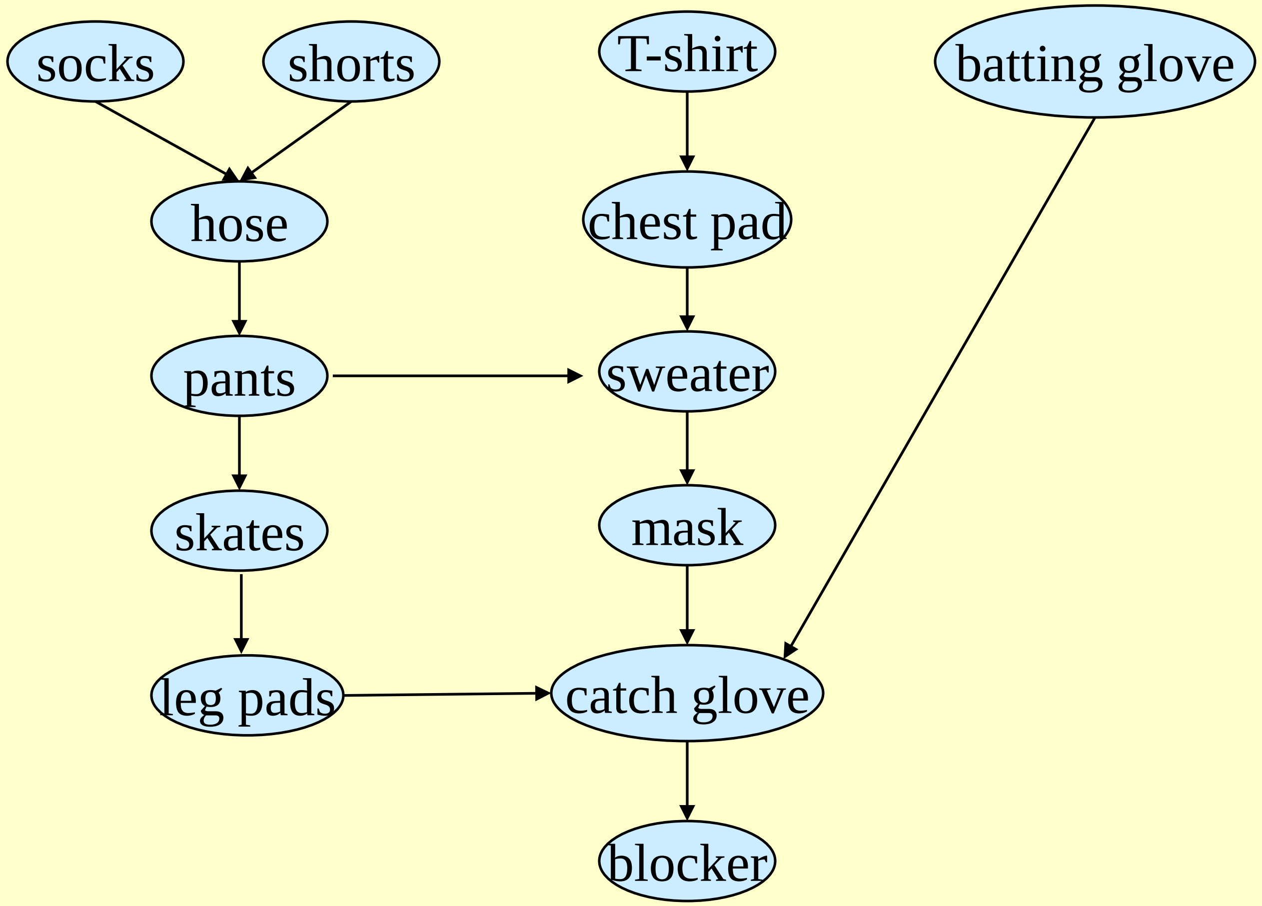
- DAGs
- Topological order
- Recognition of strongly connected components

Directed Acyclic Graph

- DAG – Directed Acyclic Graph (directed graph with no cycles)
- Used for modeling processes and structures that have a **partial order:**
  - » Let a, b, c be three elements in a set U.
  - » $a > b$ and $b > c \Rightarrow a > c$. (Transitivity)
  - » But may have a and b such that neither $a > b$ nor $b > a$.
- We can always make a **total order** (either $a > b$ or $b > a$ for all $a \neq b$) from a partial order (by imposing a relation on any two elements whose relation is not specified with the original partial order, as long as the transitivity of this partial order not violated.)

Example

DAG of dependencies for putting on goalie equipment.



Example

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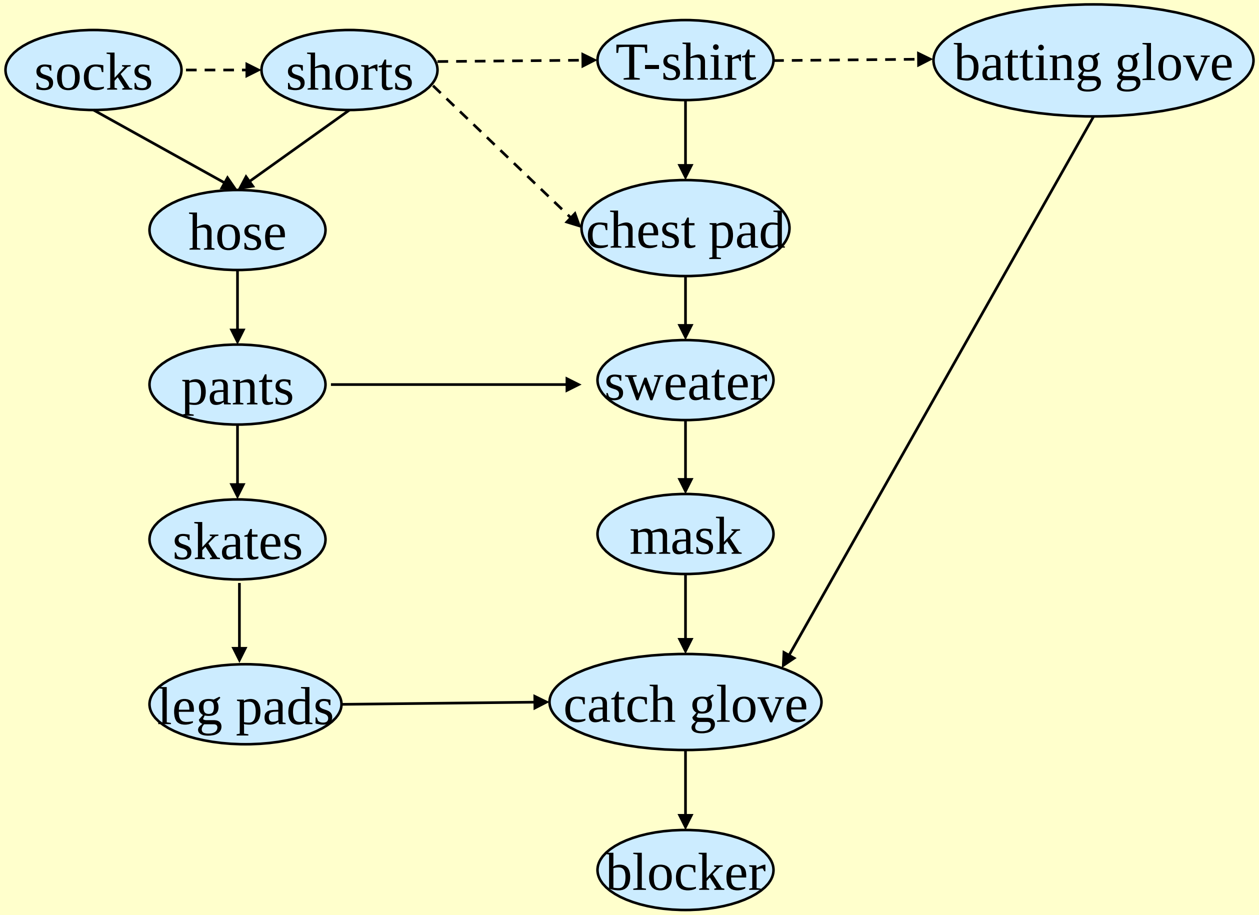


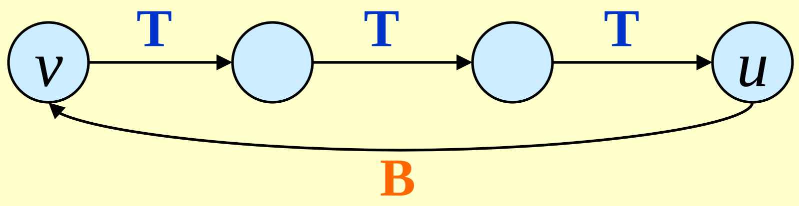
Characterizing a DAG

Lemma 22.11

A directed graph G is acyclic iff a DFS of G yields no back edges.

Proof:

- $\Rightarrow$: Show that back edge $\Rightarrow$ cycle.
  - » Suppose there is a back edge (u, v). Then v is ancestor of u in depth-first forest.
  - » Therefore, there is a path $v \leadsto u$, so $v \leadsto u \rightarrow v$ is a cycle.



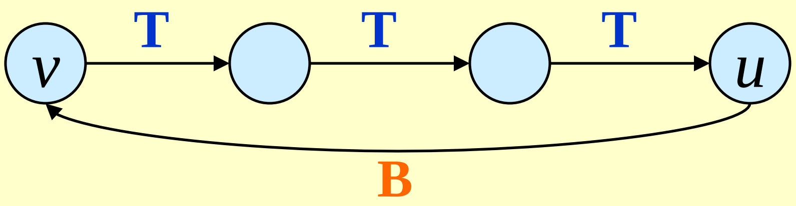
Characterizing a DAG

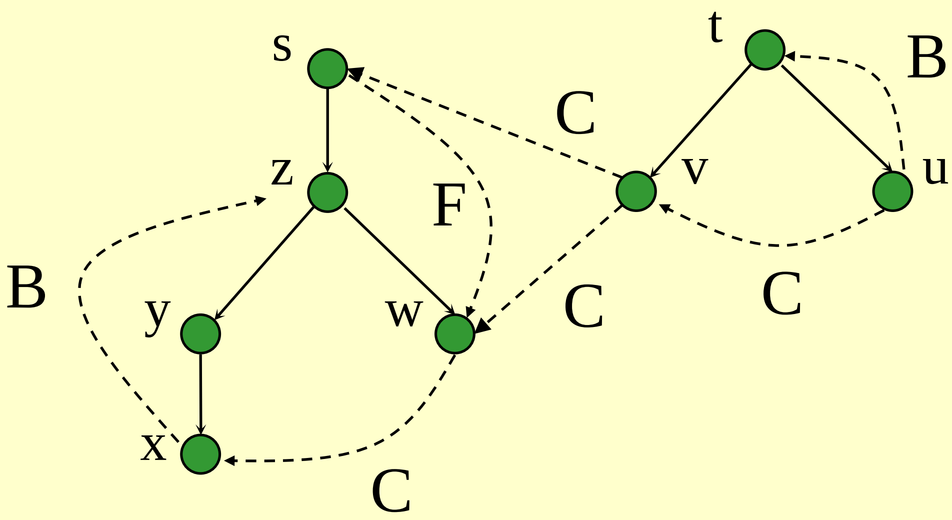
Lemma 22.11

A directed graph G is acyclic iff a DFS of G yields no back edges.

Proof (Contd.):

- $\Leftarrow$: Show that a cycle implies a back edge.
  - » c : cycle in G. v : first vertex discovered in c. (u, v) : preceding edge in c.
  - » At time $d[v]$, vertices of c form a white path $v \rightsquigarrow u$. Why?
  - » By white-path theorem, u is a descendent of v in depth-first forest.
  - » Therefore, (u, v) is a back edge.

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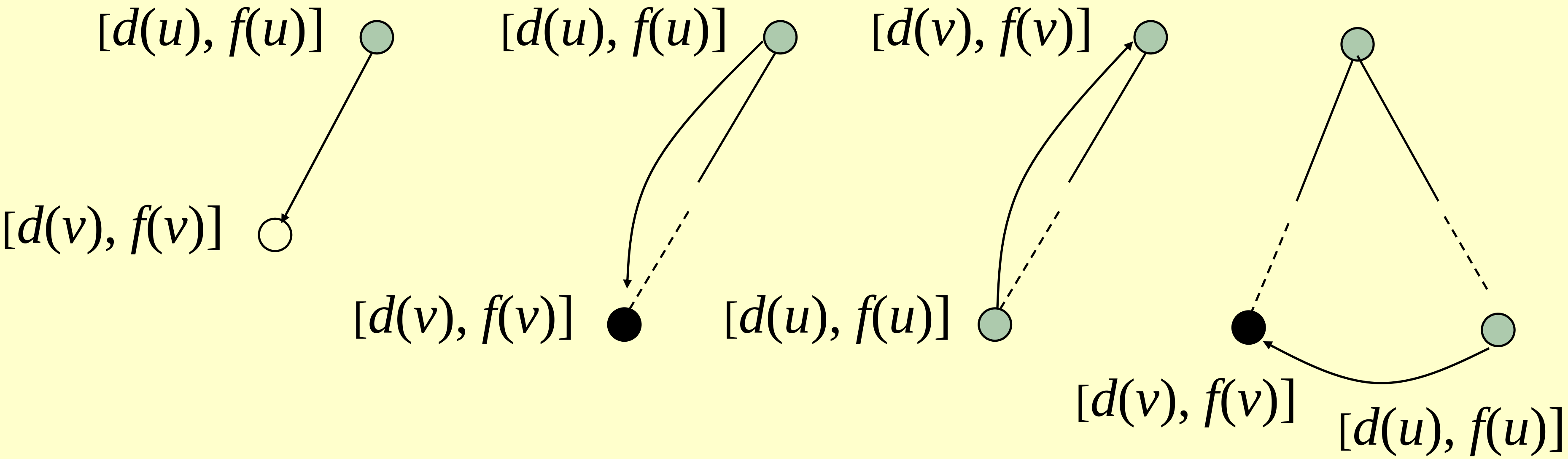
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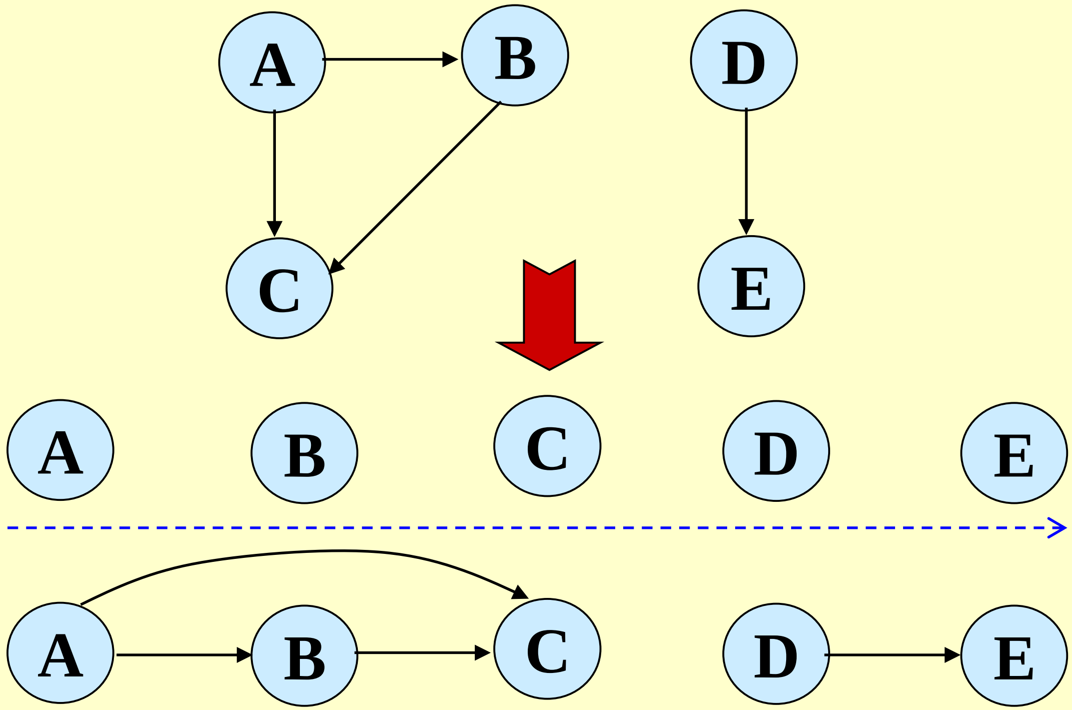
[d(u), f(u)]
[d(v), f(v)]
[d(u), f(u)]

Topological Sort

- Performed on a DAG.
- Linear ordering of the vertices of $G(V, E)$ such that if $(u, v) \in E$, then u appears somewhere before v.

Topological Sort

Sort a directed acyclic graph (DAG) by the nodes' finishing times.



Think of original DAG as a **partial order**.

By sorting, we get a **total order** that extends this partial order.

Topological Sort

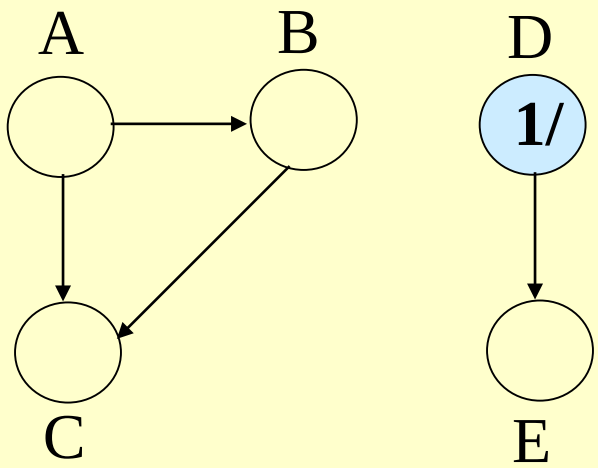
- Performed on a DAG.
- Linear ordering of the vertices of G such that if $(u, v) \in E$, then u appears somewhere before v.

Topological-Sort (G)

1. call DFS(G) to compute finishing times $f[v]$ for all $v \in V$
2. as each vertex is finished, insert it onto the front of a linked list
3. **return** the linked list of vertices

Time: $\Theta(|V| + |E|)$.

Example 1

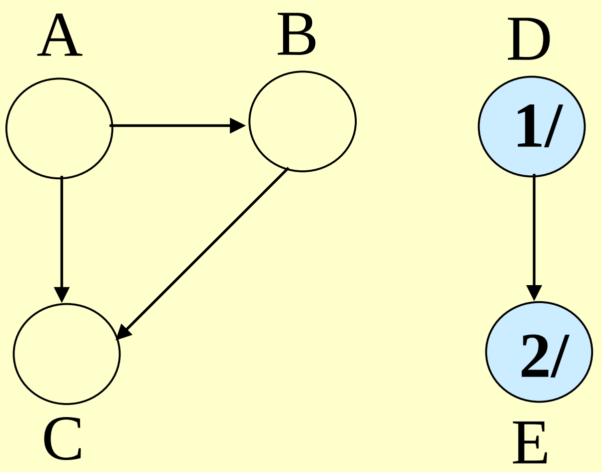


Linked List:

DFS-Visit(*u*)

```
color[u] ← GRAY   // White vertex u
                     has been discovered
time ← time + 1
d[u] ← time
for each v ∈ Adj[u]
    do if color[v] = WHITE
           then π[v] ← u
                DFS-Visit(v)
color[u] ← BLACK     // Blacken u;
                        it is finished.
f[u] ← time ← time + 1
```

Example 1



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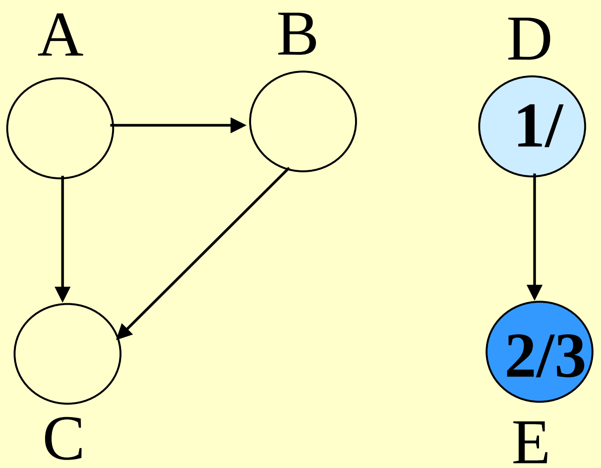
3. $d[u]$ ← *time*
4. **for** each $v \in Adj[u]$
5. **do if** *color*[v] = WHITE
6. **then** $\pi[v] \leftarrow u$
7. DFS-Visit(v)
8. *color*[u] ← BLACK // Blacken u; it is finished.
9. $f[u]$ ← *time* ← *time* + 1

Example 1



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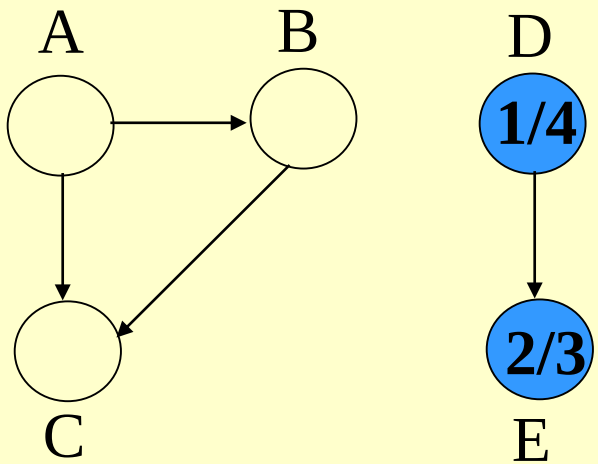

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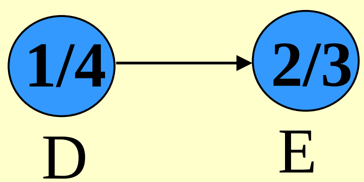

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  color[u] ← BLACK     // Blacken u;
                          it is finished.
  f[u] ← time ← time + 1
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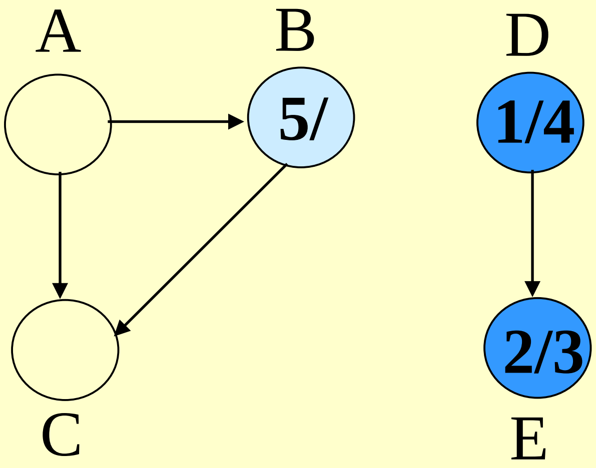

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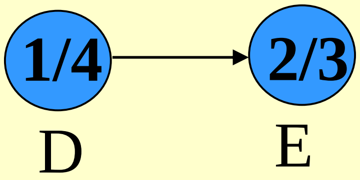
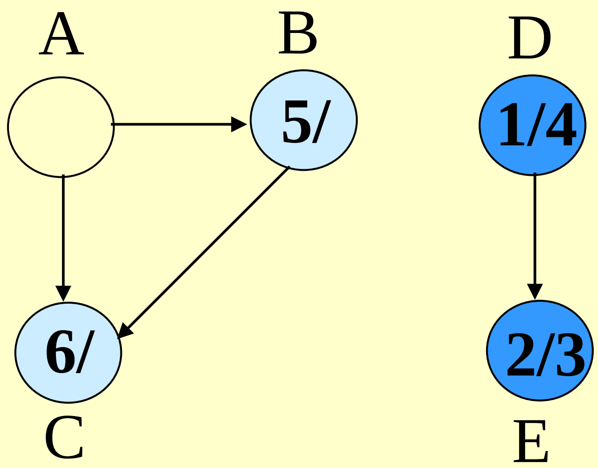


DFS(*G*)

1. **for** each vertex $u \in V[G]$
2. **do** $color[u] \leftarrow$ white
3. $\pi[u] \leftarrow$ NIL
4. $time \leftarrow 0$
5. **for** each vertex $u \in V[G]$
6. **do if** $color[u]$ = white
7. **then** DFS-Visit(u)

Example 1



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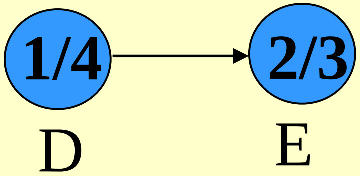

DFS-Visit(*u*)

1. *color*[*u*] ← GRAY // White vertex *u* has been discovered
2. *time* ← *time* + 1
3. *d*[*u*] ← *time*
4. **for** each $v \in Adj[u]$
5. **do if** *color*[*v*] = WHITE
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7. DFS-Visit(*v*)
8. *color*[*u*] ← BLACK // Blacken *u*; it is finished.
9. *f*[*u*] ← *time* ← *time* + 1

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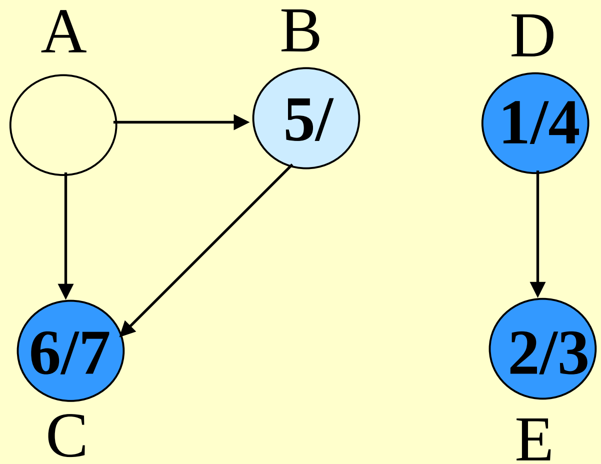

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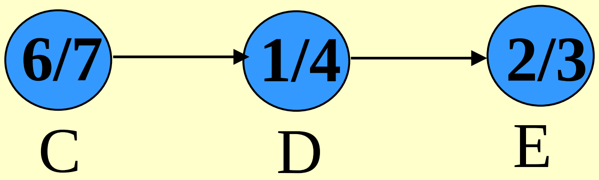

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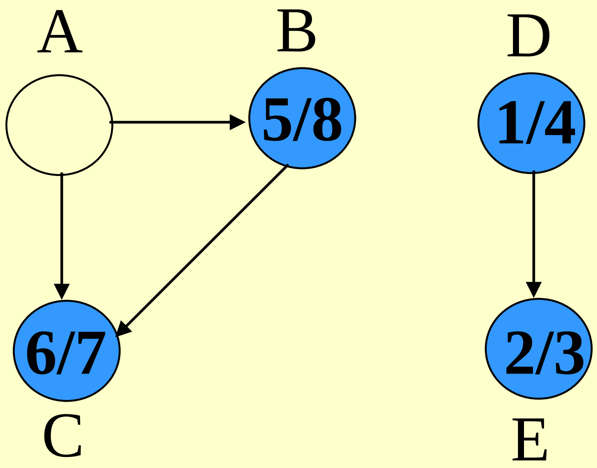

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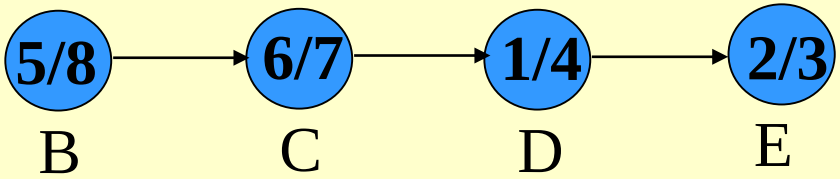

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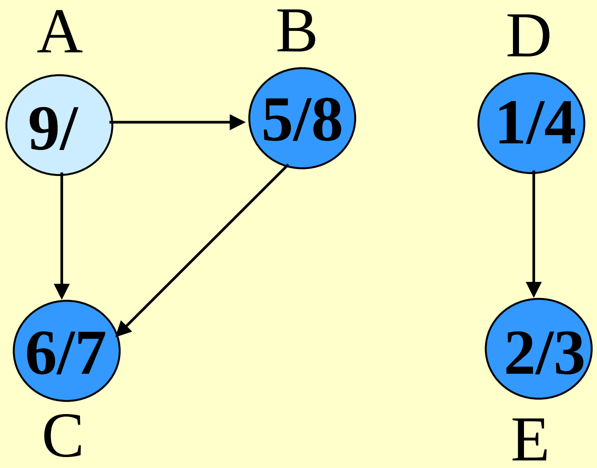

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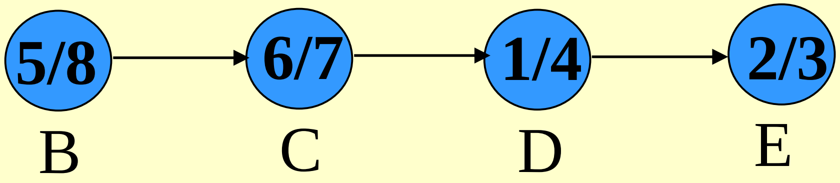

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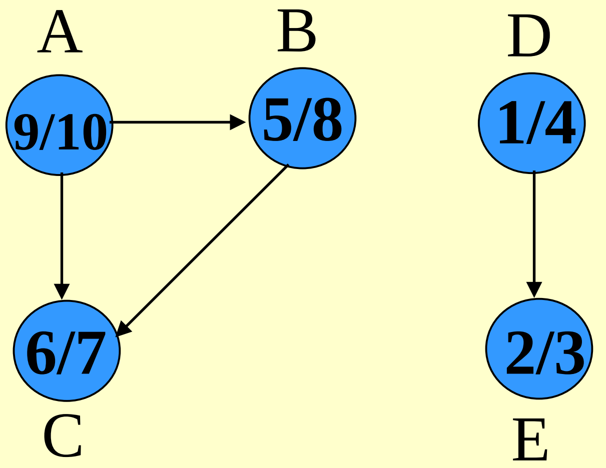

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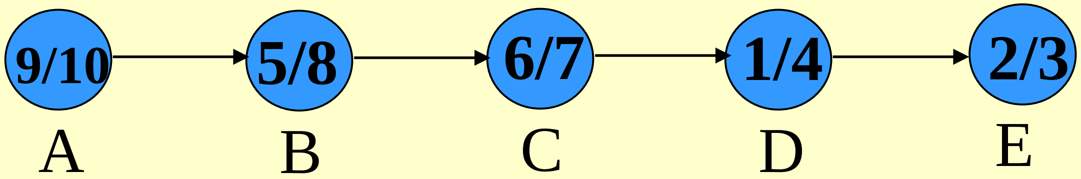
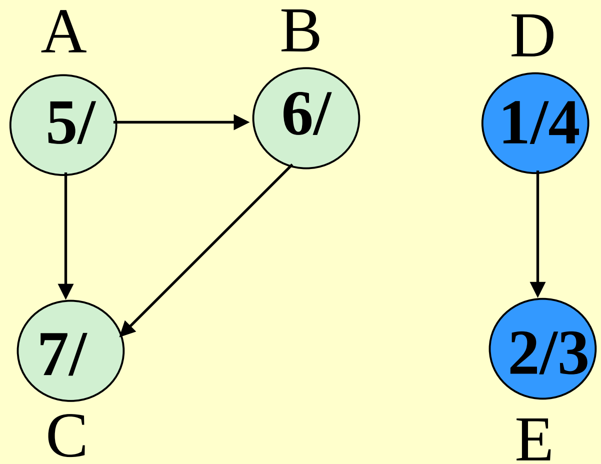


Example 2

Access the nodes in different way:



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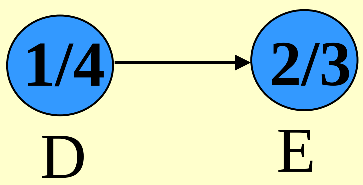
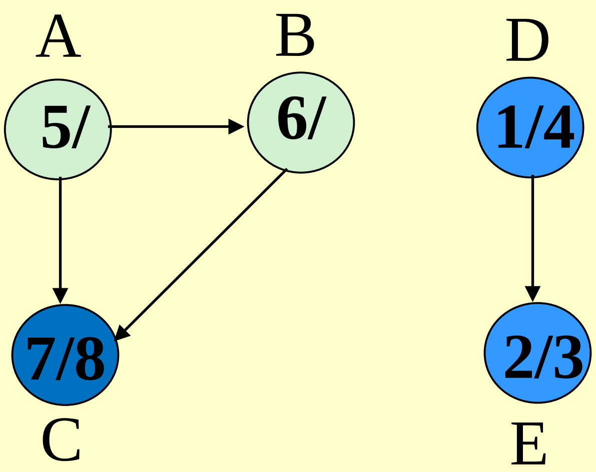


Example 2



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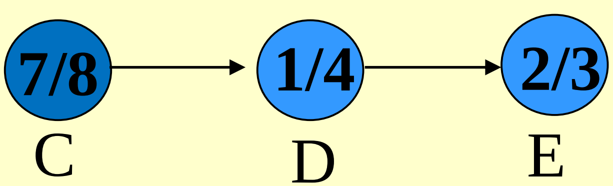

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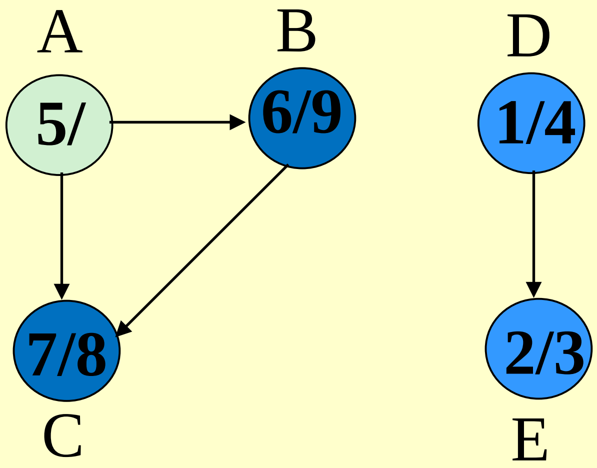

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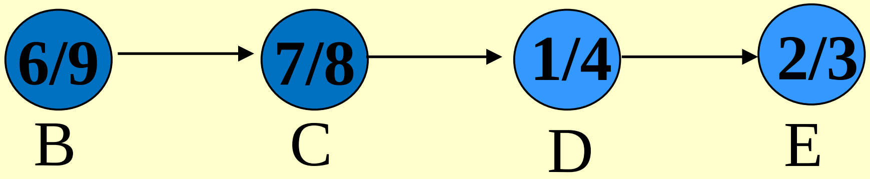

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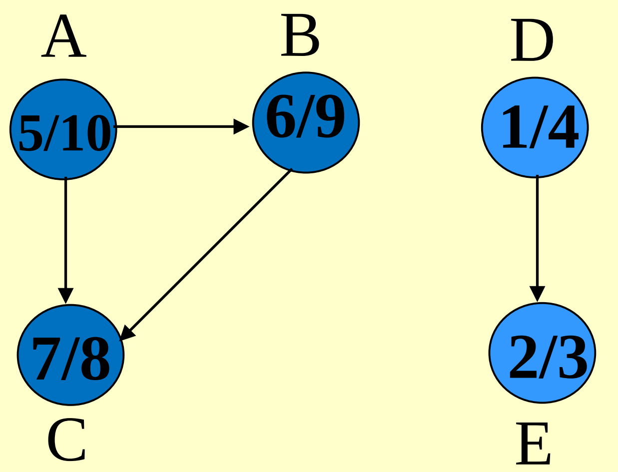

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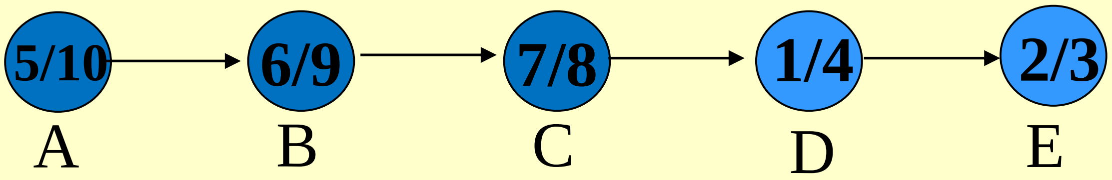
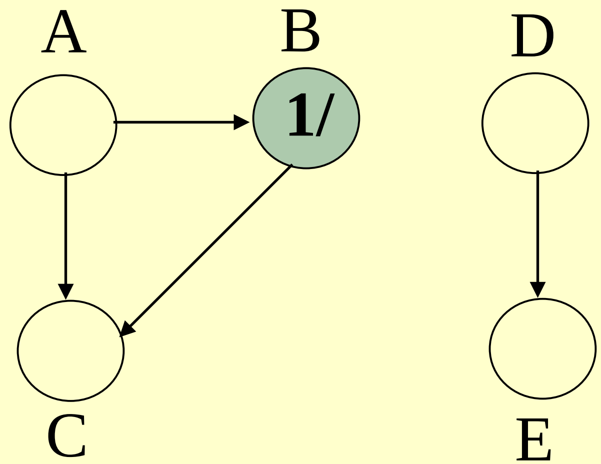


Example 3



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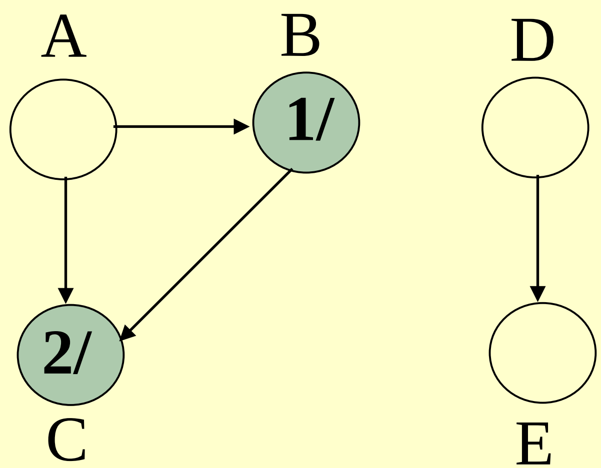

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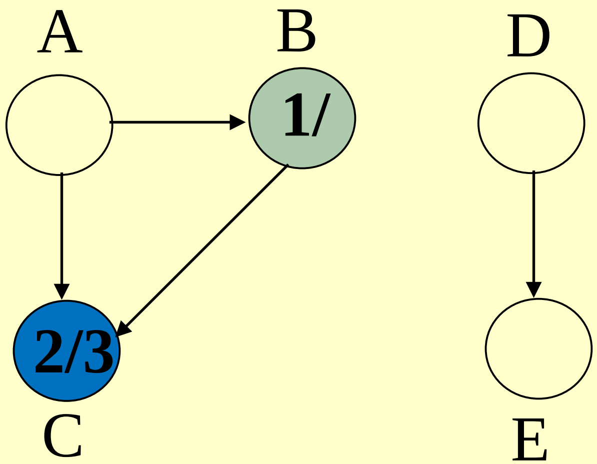

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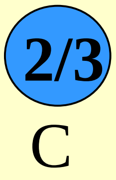

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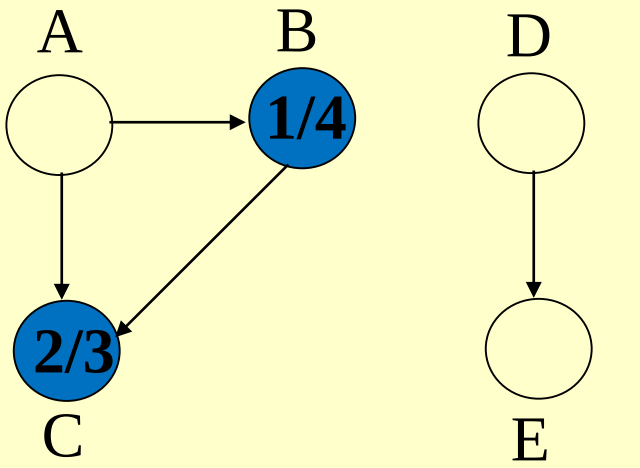

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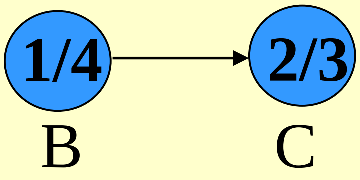

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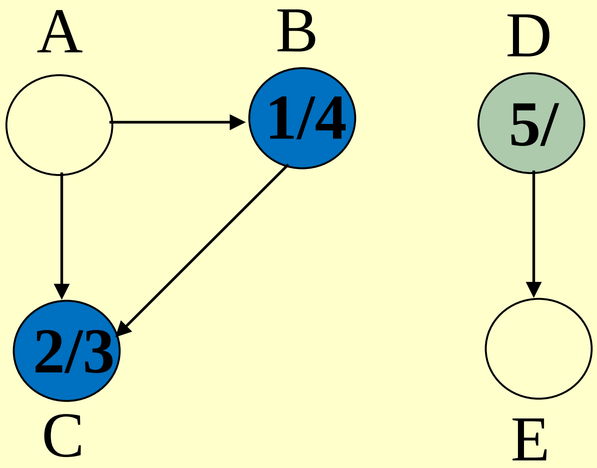

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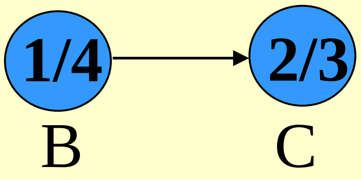

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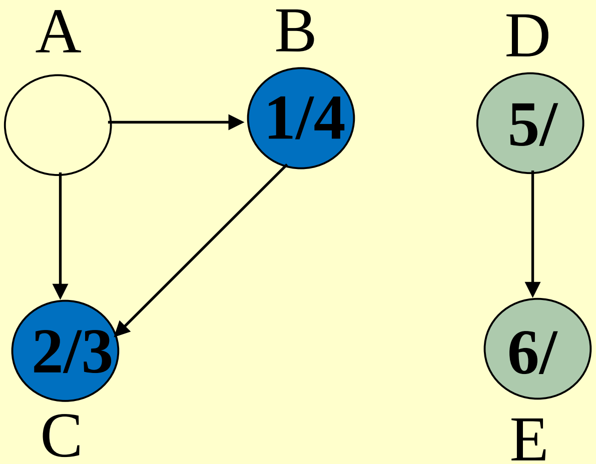

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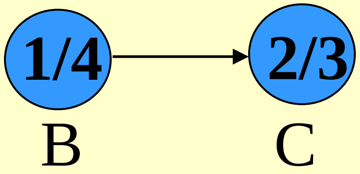

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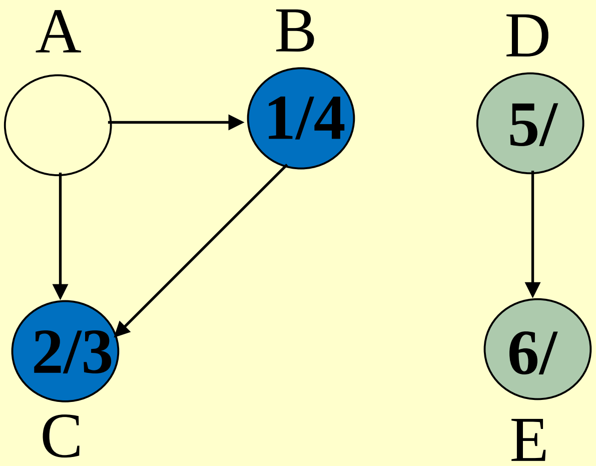

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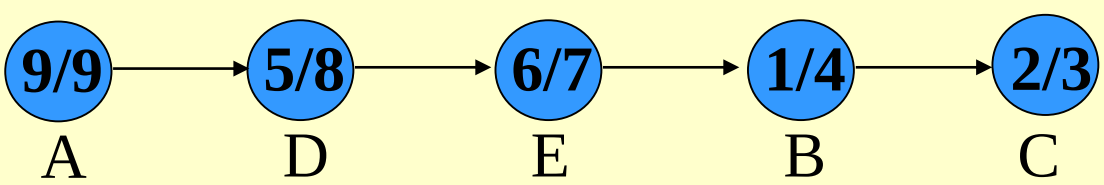
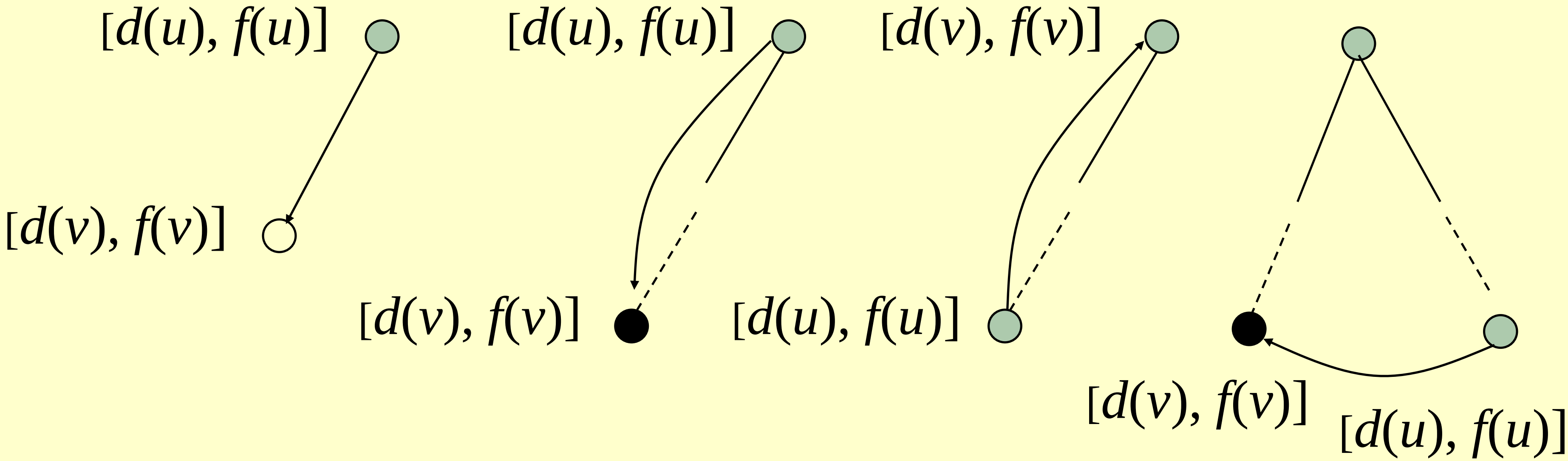


Correctness Proof

- Just need to show if $(u, v) \in E$, then $f[u] > f[v]$.
- When we explore (u, v), what are the colors of u and v?
  - » u is gray.
  - » Is v white?
    - Then becomes descendant of u.
    - By parenthesis theorem, $d[u] < d[v] < \underline{f[v] < f[u]}$.
  - » Is v black?
    - Then v is already finished.
    - Since we're exploring (u, v), we have not yet finished u.
    - Therefore, $f[v] < f[u]$.
  - » Is v gray, too?
    - No.
    - because then v would be ancestor of $u \Rightarrow (u, v)$ is a back edge.
    - $\Rightarrow$ contradiction of Lemma 22.11 (dag has no back edges).



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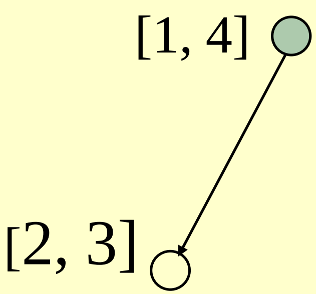
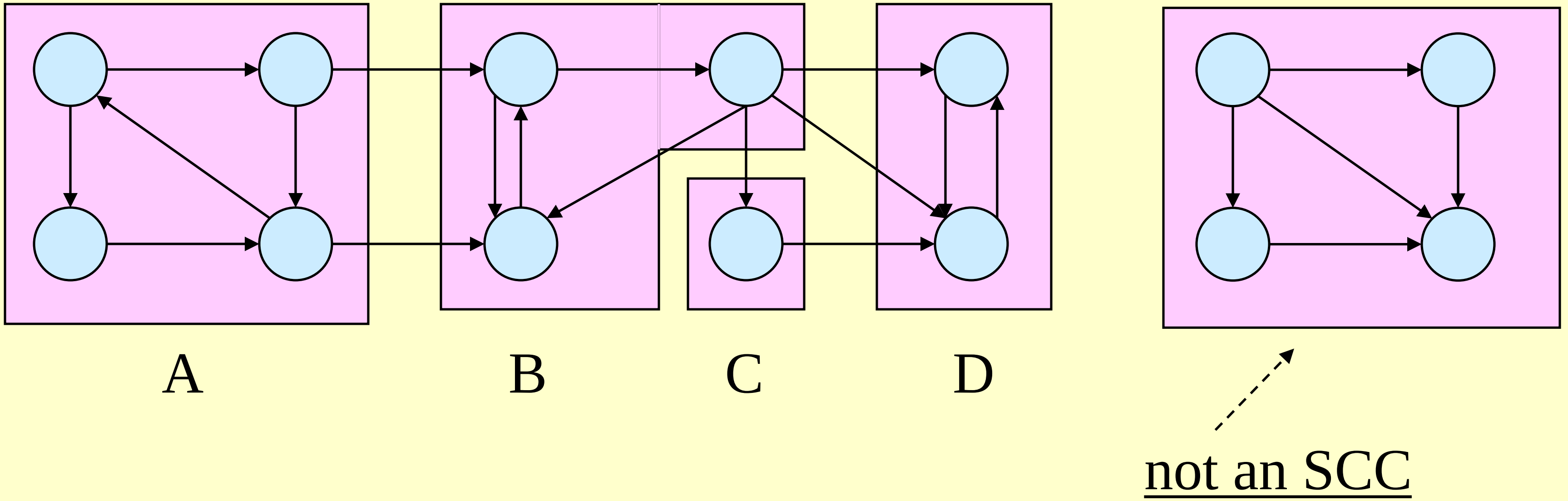


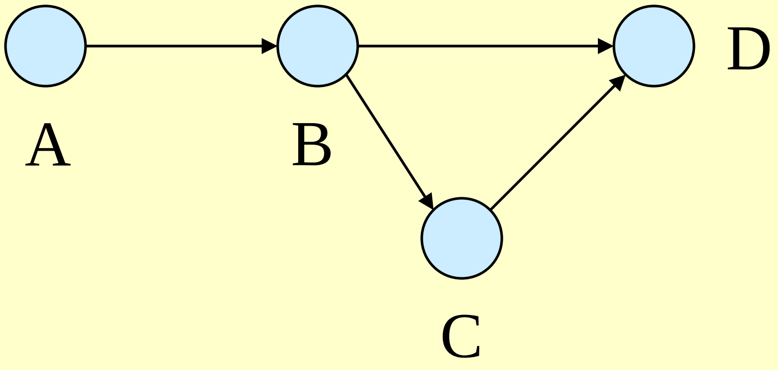
Strongly Connected Components

- G is strongly connected if every pair (u, v) of vertices in G is reachable from one another.
- A **strongly connected component** (***SCC***) of G is a maximal set of vertices $C \subseteq V$ such that for all $u, v \in C$, both $u \rightsquigarrow v$ and $v \rightsquigarrow u$ exist.



Component Graph

- $G^{SCC} = (V^{SCC}, E^{SCC})$.
- V^{SCC} has one vertex for each SCC in G.
- E^{SCC} has an edge if there's an edge between the corresponding SCC's in G.
- G^{SCC} for the example considered:



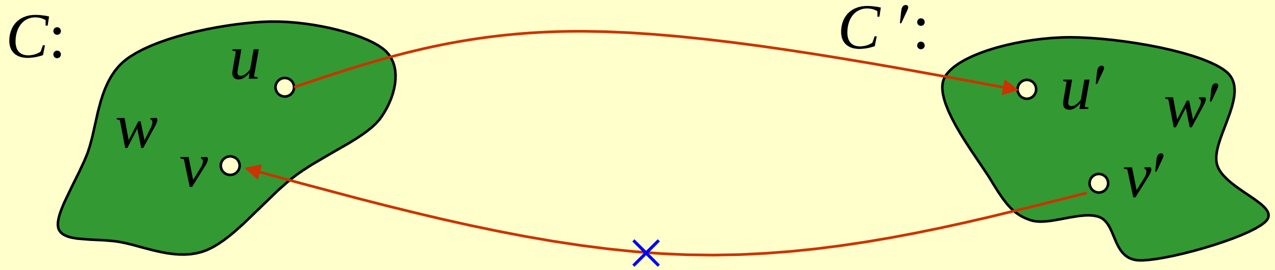
G^{SCC} is a DAG

Lemma 22.13

Let C and C' be distinct SCC's in G, let $u, v \in C$, $u', v' \in C'$, and suppose there is a path $u \leadsto u'$ in G. Then there cannot also be a path $v' \leadsto v$ in G.

Proof:

- Suppose there is a path $v' \leadsto v$ in G.
- Then there are paths $u \leadsto u' \leadsto v'$ and $v' \leadsto v \leadsto u$ in G.
- Therefore, u and v' are reachable from each other, so they are not in separate SCC's.



Transpose of a Directed Graph

- G^T = **transpose** of directed G.
  - » $G^T = (V, E^T)$, $E^T = \{(u, v) : (v, u) \in E\}$.
  - » G^T is G with all edges reversed.
- Can create G^T in $\Theta(|V| + |E|)$ time if using adjacency lists.
- G and G^T have the *same* SCC's. (u and v are reachable from each other in G if and only if reachable from each other in G^T.)

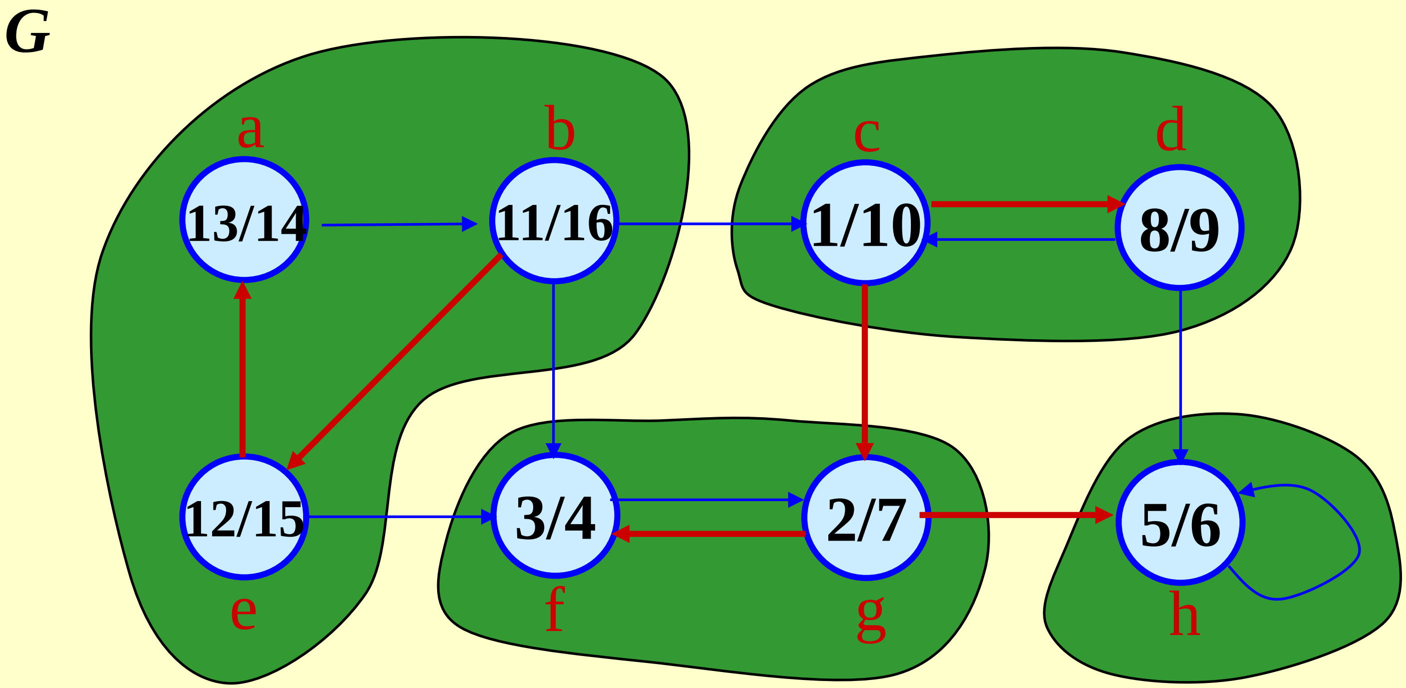
Algorithm to determine SCCs

SCC(G)

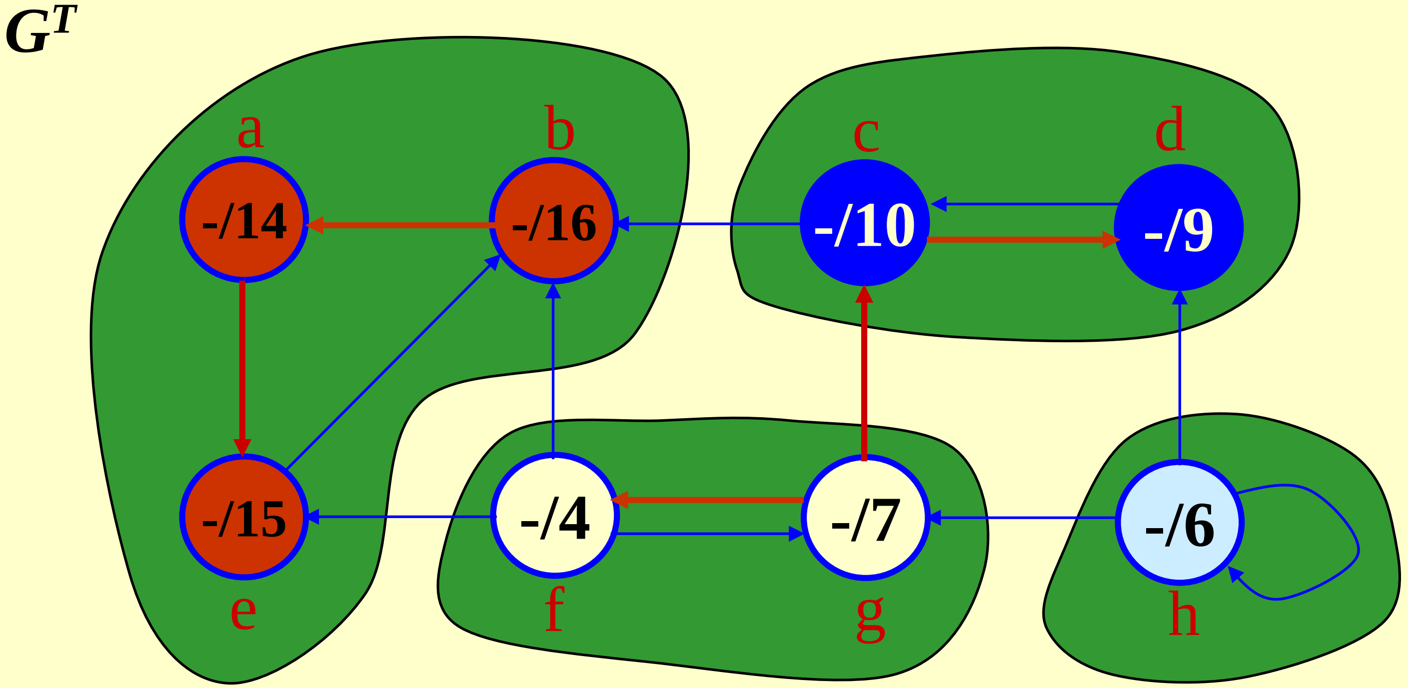
1. call DFS(G) to compute finishing times $f[u]$ for all u
2. compute G^T
3. call DFS(G^T), but in the main loop, consider vertices in order of decreasing $f[u]$ (as computed in the first DFS)
4. output the vertices in each tree of the depth-first forest formed in the second DFS as a separate SCC

Time: $\Theta(|V| + |E|)$.

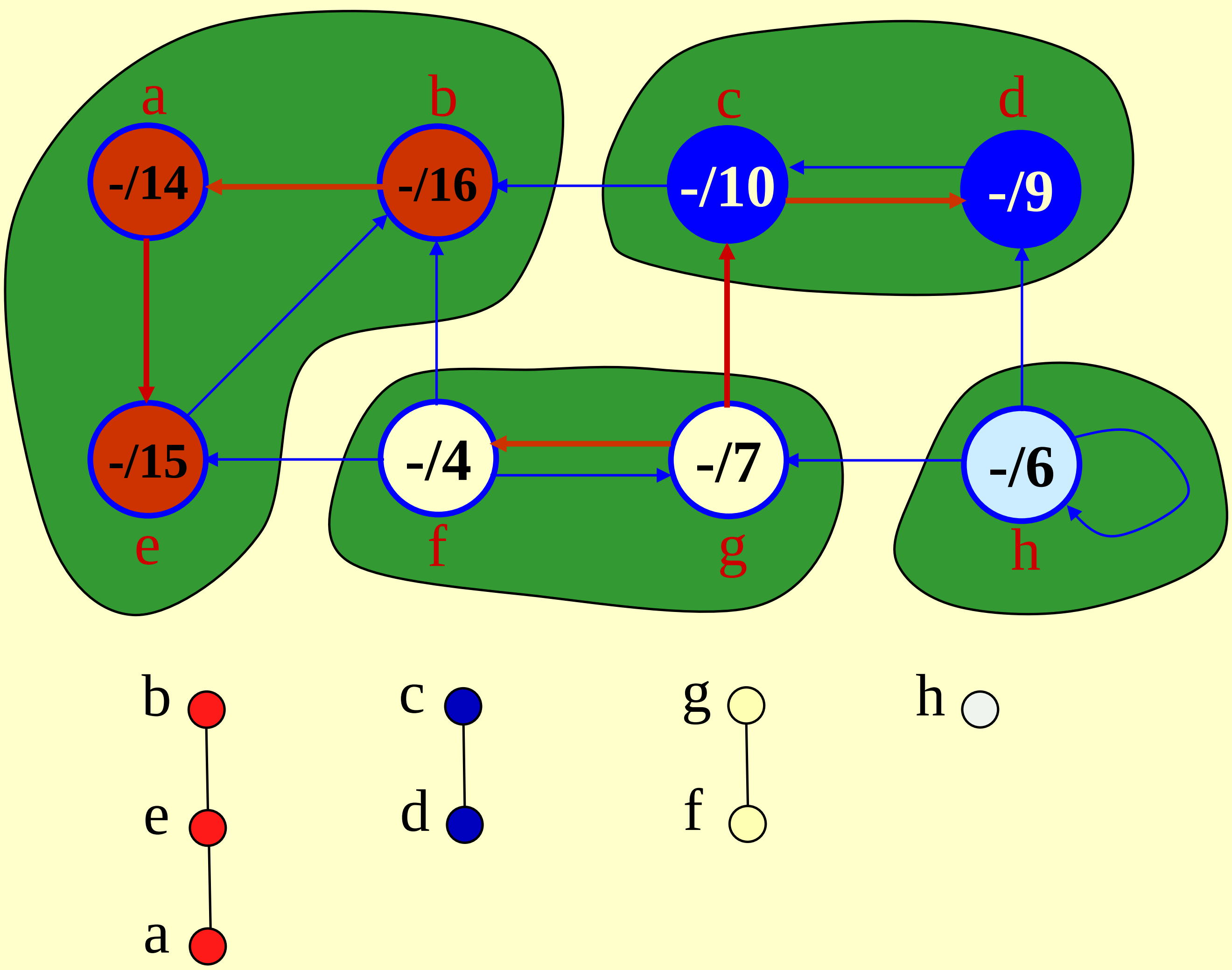
Example



Example



Example



How does it work?

- **Idea:**
  - » By considering vertices in second DFS in decreasing order of finishing times from first DFS, we are visiting vertices of the component graph in topologically sorted order.
  - » Because we are running DFS on G^T, we will not be visiting any v from a u, where v and u are in different components.
- **Notation:**
  - » $d[u]$ and $f[u]$ always refer to *first* DFS.
  - » Extend notation for d and f to sets of vertices $U \subseteq V$:
  - » $d(U) = \min_{u \in U}\{d[u]\}$ (earliest discovery time)
  - » $f(U) = \max_{u \in U}\{f[u]\}$ (latest finishing time)

SCCs and DFS finishing times

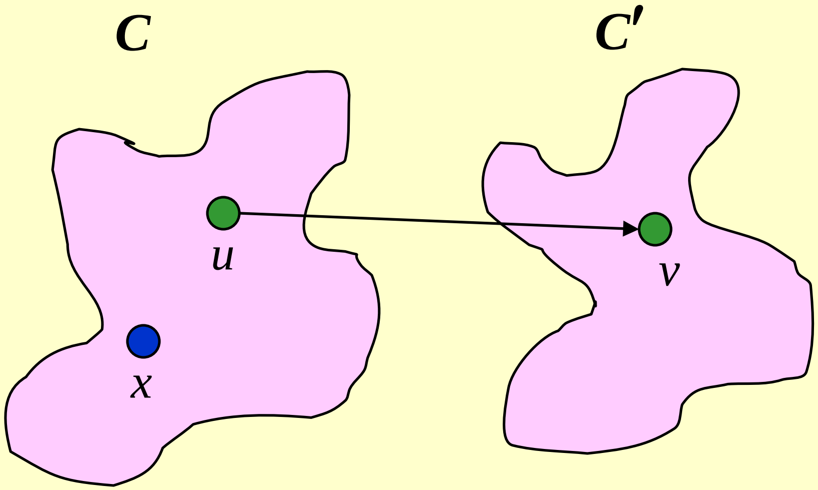
Lemma 22.14

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Proof:

- Case 1: $d(C) < d(C')$
  - » Let x be the first vertex discovered in C.
  - » At time $d[x]$, all vertices in C and C' are white. Thus, there exist paths of white vertices from x to all vertices in C and C'.
  - » By the white-path theorem, all vertices in C and C' are descendants of x in depth-first tree.
  - » By the parenthesis theorem, $f[x] = f(C) > f(C')$.

$$d(x) < d(v) < f(v) < f(x)$$



$$d(C) = \min_{u \in C}\{d[u]\})$$
$$f(C) = \max_{u \in C}\{f[u]\}$$

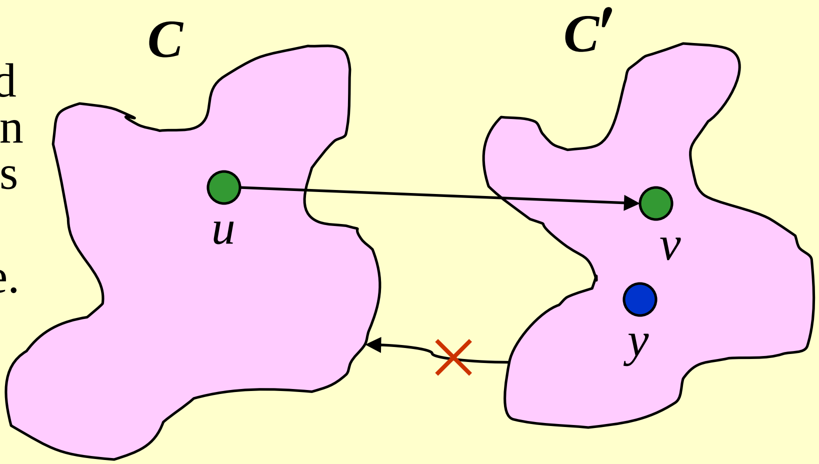
SCCs and DFS finishing times

Lemma 22.14

Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E$ such that $u \in C$ and $v \in C'$. Then $f(C) > f(C')$.

Proof:

- Case 2: $d(C) > d(C')$
  - » Let y be the first vertex discovered in C'.
  - » At time $d[y]$, all vertices in C' are white and there is a white path from y to each vertex in C' $\Rightarrow$ all vertices in C' become descendants of y. Again, $f[y] = f(C')$.
  - » At time $d[y]$, all vertices in C are also white.
  - » By earlier lemma, since there is an edge (u, v), we cannot have a path from C' to C.
  - » So no vertex in C is reachable from y.
  - » Therefore, at time $f[y]$, all vertices in C are still white.
  - » Therefore, for all $v \in C$, $f[v] > f[y]$, which implies that $f(C) > f(C')$.



$d(C) = \min_{u\in C}\{d[u]\})$

$f(C) = \max_{u\in C}\{f[u]\}$

SCCs and DFS finishing times

Corollary 22.15
Let C and C' be distinct SCC's in $G = (V, E)$. Suppose there is an edge $(u, v) \in E^T$, where $u \in C$ and $v \in C'$. Then $f(C) < f(C')$.

Proof:

- $(u, v) \in E^T \Rightarrow (v, u) \in E$.
- Since SCC's of G and G^T are the same, $f(C') > f(C)$, by Lemma 22.14.

Correctness of SCC

- When we do the second DFS, on G^T, start with SCC C such that $f(C)$ is maximum.
  - » The second DFS starts from some $x \in C$, and it visits all vertices in C.
  - » Corollary 22.15 says that since $f(C) > f(C')$ for all $C \neq C'$, there are no edges from C to C' in G^T.
  - » Therefore, DFS will visit *only* vertices in C.
  - » Which means that the depth-first tree rooted at x contains *exactly* the vertices of C.

Correctness of SCC

- The next root chosen in the second DFS is in SCC C' such that $f(C')$ is maximum over all SCC's other than C.
  - » DFS visits all vertices in C', but the only edges out of C' go to C, *which we've already visited*.
  - » Therefore, the only tree edges will be to vertices in C'.
- We can continue the process.
- Each time we choose a root for the second DFS, it can reach only
  - » vertices in its SCC—get tree edges to these,
  - » vertices in SCC's *already visited* in second DFS—get *no* tree edges to these.