String Matching

- String matching problem
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  - suffix, suffix function
  - prefix, prefix function
  - Knuth-Morris-Pratt algorithm

Chapter 32: String Matching

String-matching problem

1. **Text: an array $T[1 .. n]$ containing n characters drawn from a finite alphabet Σ**

   (for instance, $\Sigma = \{0, 1\}$ or $\Sigma = \{a, b, …, z\}$.)

   Pattern: an array $P[1 .. m]$ ($m \leq n$)
2. **Finding all occurrences of a pattern in a text is a problem that arises frequently in text-editing programs.**

Definition

We say that pattern P occurs with shift s in text T (or, equivalently, that pattern P occurs beginning at position $s + 1$ in text T)

if $0 \leq s \leq n - m$ and

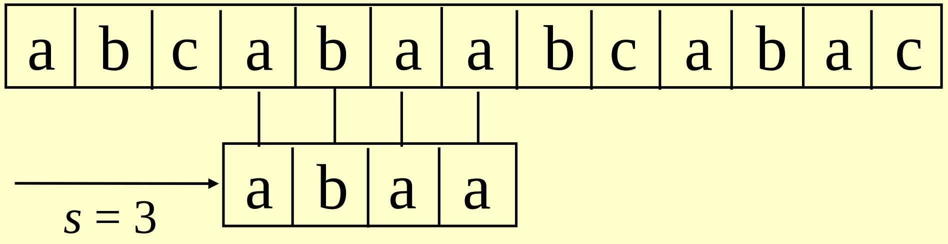
$T[s + 1 .. s + m] = P[1 .. m]$

(i.e., if $T[s + j] = P[j]$ for $1 \leq j \leq m$).

Valid shift s – if P occurs with shift s in T. Otherwise, s is an invalid shift.

text T:

pattern P:



We will find all the valid shifts.

■ Naïve algorithm

```
Naïve-String-Matcher($T$, $P$)
1. $n \leftarrow length[T]$
2. $m \leftarrow length[P]$
3. **for** $s \leftarrow 0$ **to** $n - m$
4.      **do if** $T[s + 1 .. s + m] = P[1 .. m]$
5.           **then** print “Pattern occurs with shift” $s$
```

Obviously, the time complexity of this algorithm is bounded by O(nm).

In the following, we will discuss Knuth-Morris-Pratt algorithm, which needs only O($n + m$) time.

■ Finite automata

A finite automaton M is a 5-tuple $(Q, q_0, A, \Sigma, \delta)$, where

Q - a finite set of states

q_0 - the start state

$A \subseteq Q$ – a distinguished set of accepting states

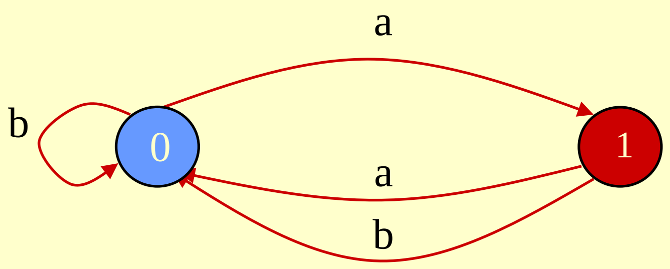
Σ - a finite input alphabet

δ - a function from $Q \times \Sigma$ into Q, called the transition function of M.

Example: $Q = \{0, 1\}$, $q_0 = 0$, $A = \{1\}$, $\Sigma = \{a, b\}$

$\delta(0, a) = 1$, $\delta(0, b) = 0$, $\delta(1, a) = 0$, $\delta(1, b) = 0$.

	input	
state	a	b
0	1	0
1	0	0



String-matching automata for patterns

Σ^* - the set of all finite-length strings formed using characters from the alphabet Σ

ε - zero-length *empty string*

$|x|$ - the length of string x

xy - the concatenation of two strings x and y, which has length $|x| + |y|$ and consists of the characters from x followed by the characters from y

prefix – a string w is a prefix of a string x, denoted w ◉ x, if $x = wy$ for some $y \in \Sigma^*$.

suffix – a string w is a suffix of a string x, denoted w ▪ x, if $x = yw$ for some $y \in \Sigma^*$.

Example: ab ◉ abcca. cca ▪ abcca.

■ String-matching automata for patterns

- P_k - $P[1\,..\,k]$ ($k \leq m$), a prefix of $P[1\,..\,m]$

  suffix function σ - a mapping from Σ^* to $\{0, 1, \ldots, m\}$ such that $\sigma(x)$ is the length of the longest prefix of P that is a suffix of x:

  $$\sigma(x) = \max\{k: P_k \blacksquare x\}.$$

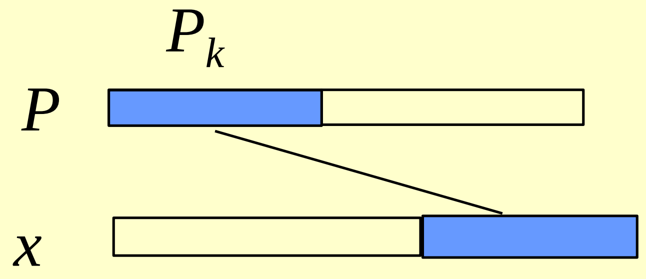
  Note that $P_0 = \varepsilon$ is a suffix of every string.
- Example

  P = ab

  We have $\sigma(\varepsilon) = 0$

  $\sigma(\text{ccac}\underline{\text{a}}) = 1$ $\qquad$ $P = \underline{\text{a}}\text{b}$

  $\sigma(\text{cc}\underline{\text{ab}}) = 2$ $\qquad$ $P = \underline{\text{ab}}$



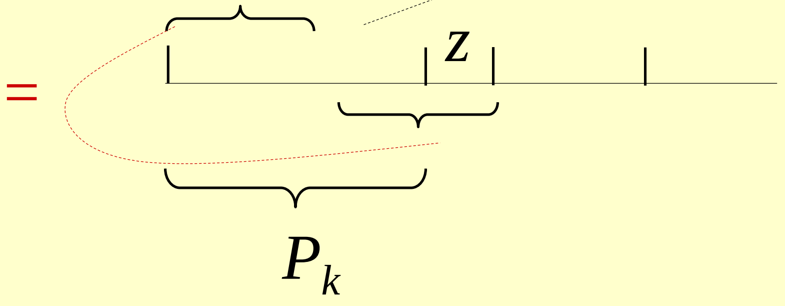
■ String-matching automata for a pattern

For a pattern $P[1 .. m]$, its string-matching automaton can be constructed as follows.

1. The state set Q is $\{0, 1, \ldots, m\}$. The start state q_0 is state 0, and state m is the only accepting state. Σ contains all the characters in P.
2. The transition function δ is defined by the following equation, for any state k and character z:

$$\delta(k, z) = \sigma(P_k z)$$

$P = \underline{ab}cad \ldots \ldots$



$$\delta(4, b) = \sigma(P_4 b) = \sigma(abc\underline{ab}) = 2$$

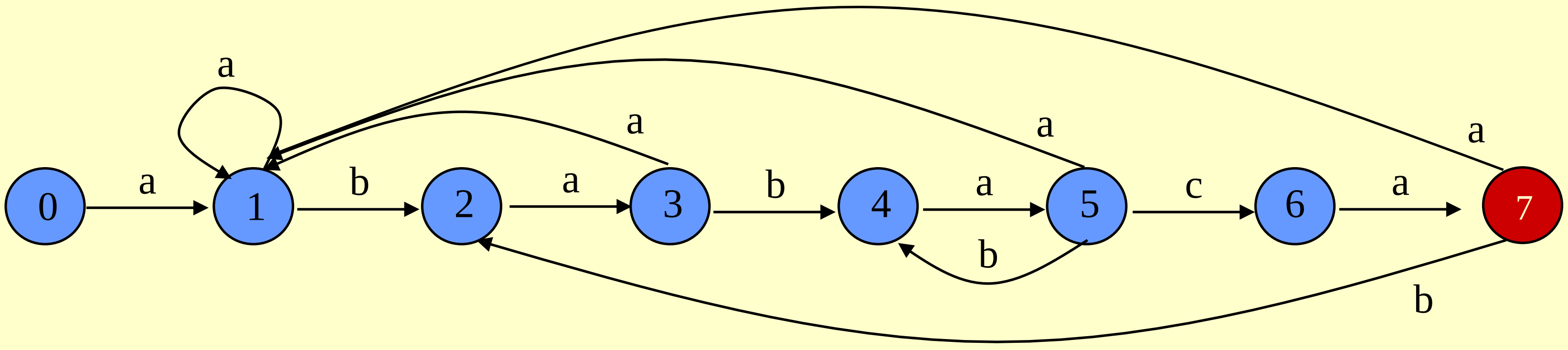
$$\delta(4, d) = \sigma(P_4 d) = \sigma(\underline{abcad}) = 5$$

String-matching automata for patterns

- Example

P = ababaca

$\delta(k, z) = \sigma(P_k z)$


$$\begin{tikzcd}
0 \arrow[r, "a"] & 1 \arrow[r, "b"] \arrow[loop above, "a"] & 2 \arrow[r, "a"] & 3 \arrow[r, "b"] \arrow[ll, bend right, "a"'] & 4 \arrow[r, "a"] & 5 \arrow[r, "c"] \arrow[l, bend left, "b"] \arrow[llll, bend right, "a"'] & 6 \arrow[r, "a"] & 7 \arrow[llllll, bend right, "a"'] \arrow[lllll, bend left, "b"]
\end{tikzcd}$$

P	a	b	a	b	a	c	a	
State	0	1	2	3	4	5	6	7

transition function

input	0	1	2	3	4	5	6	7
a	1	1	3	1	5	1	7	1
b	0	2	0	4	0	4	0	2
c	0	0	0	0	0	6	0	0

Assume that $P_0 = \varepsilon$.

■ String-matching automata for patterns

- Example

P = ababaca

$\delta(k, z) = \sigma(P_k z)$

P	a	b	a	b	a	c	a	
State	0	1	2	3	4	5	6	7

transition function

input								
a	1	1	3	1	5	1	7	1
b	0	2	0	4	0	4	0	2
c	0	0	0	0	0	6	0	0

Assume that $P_0 = \varepsilon$.

$\delta(0, \text{a}) = \sigma(P_0\text{a}) = \sigma(\text{a}) = 1$ $\quad$ $\delta(1, \text{a}) = \sigma(P_1\text{a}) = \sigma(\text{aa}) = 1$

$\delta(0, \text{b}) = \sigma(P_0\text{b}) = \sigma(\text{b}) = 0$ $\quad$ $\delta(1, \text{b}) = \sigma(P_1\text{b}) = \sigma(\text{ab}) = 2$ $\quad$ … …

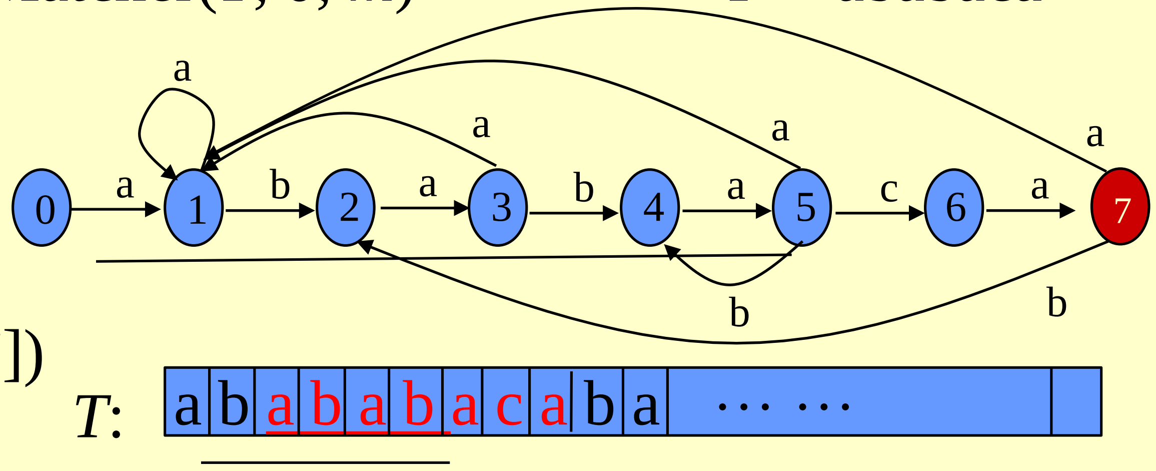
$\delta(0, \text{c}) = \sigma(P_0\text{c}) = \sigma(\text{c}) = 0$ $\quad$ $\delta(1, \text{c}) = \sigma(P_1\text{c}) = \sigma(\text{ac}) = 0$

■Finite-Automaton-Matcher

- String matching by using the finite automaton

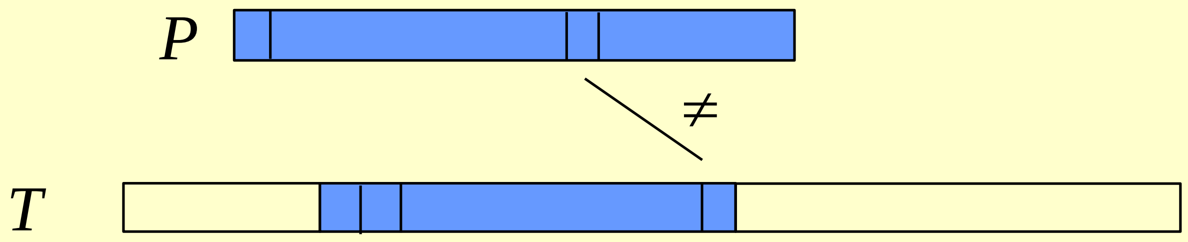
```
Finite-Automaton-Matcher($T$, $\delta$, $m$)
1. $n \leftarrow length[T]$
2. $q \leftarrow 0$
3. for $i \leftarrow 1$ to $n$
4.   do $q \leftarrow \delta(q, T[i])$
5.      if $q = m$
6.      then print "pattern occurs with shift" $i - m$
```

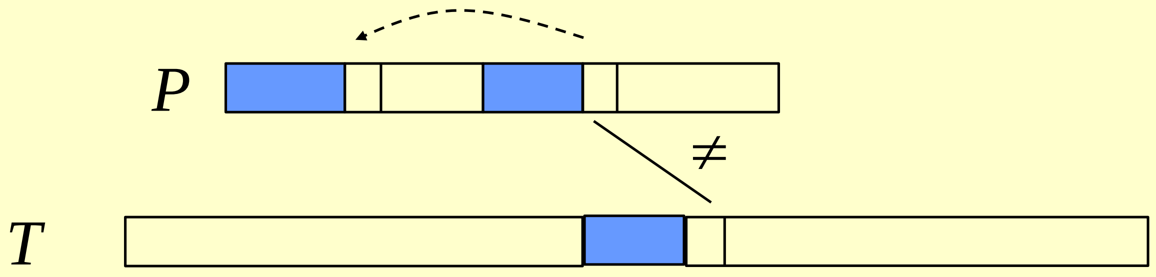
P = ababaca



If the finite automaton is available, the algorithm needs only O(n + m) time.

Why should the string matching automaton be precomputed?



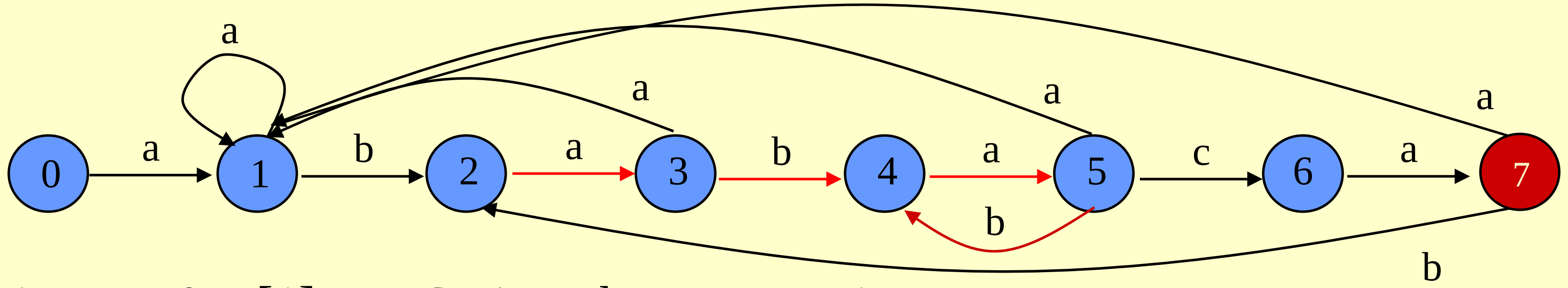


■ Finite-Automaton-Matcher

- Example

P = ababaca, T = abababacaba

$\delta(k, z) = \sigma(P_k z)$



step 1: $q = 0$, $T[1]$ = a. Go into the state $q = 1$.
step 2: $q = 1$, $T[2]$ = b. Go into the state $q = 2$.
step 3: $q = 2$, $T[3]$ = a. Go into the state $q = 3$.
step 4: $q = 3$, $T[4]$ = b. Go into the state $q = 4$.
step 5: $q = 4$, $T[5]$ = a. Go into the state $q = 5$.
step 6: $q = 5$, $T[6]$ = b. Go into the state $q = 4$.
step 7: $q = 4$, $T[7]$ = a. Go into the state $q = 5$.
step 8: $q = 5$, $T[8]$ = c. Go into the state $q = 6$.
step 9: $q = 6$, $T[9]$ = a. Go into the state $q = 7$.

P	*a*	*b*	*a*	*b*	*a*	*c*	*a*	
State	0	1	2	3	4	5	6	7

a	1	1	3	1	5	1	7	1
b	0	2	0	4	0	4	0	2
c	0	0	0	0	0	6	0	0

■ Knuth-Morris-Pratt algorithm

- Dynamic computation of the transition function δ

We needn't compute δ altogether, but using an auxiliary function π, called a *prefix function*, to calculate δ–values "on the fly".

prefix function π - a mapping from {1, …, m} to {0, 1, …, m} such that

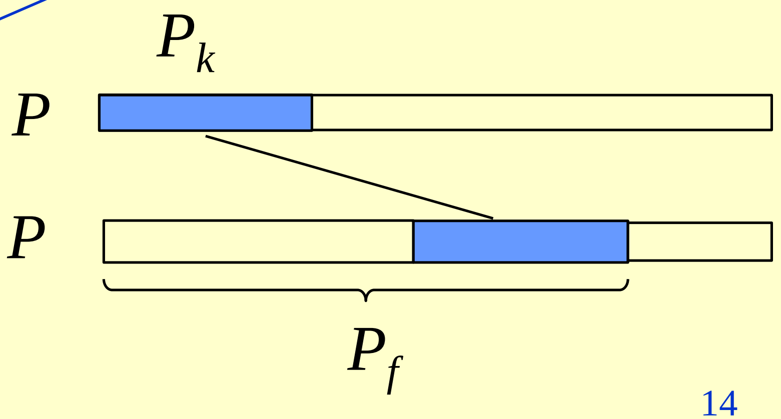
$$\pi(f) = \max\{k: k < f, P_k \blacksquare P_f\}.$$

comparison with suffix function:

$$\sigma(x) = \max\{k: P_k \blacksquare x\}$$

$\sigma(P_k z) = \delta(k, z)$

$z \in \Sigma$



■ Knuth-Morris-Pratt algorithm

- Example

P = ababababca

P = ababaca

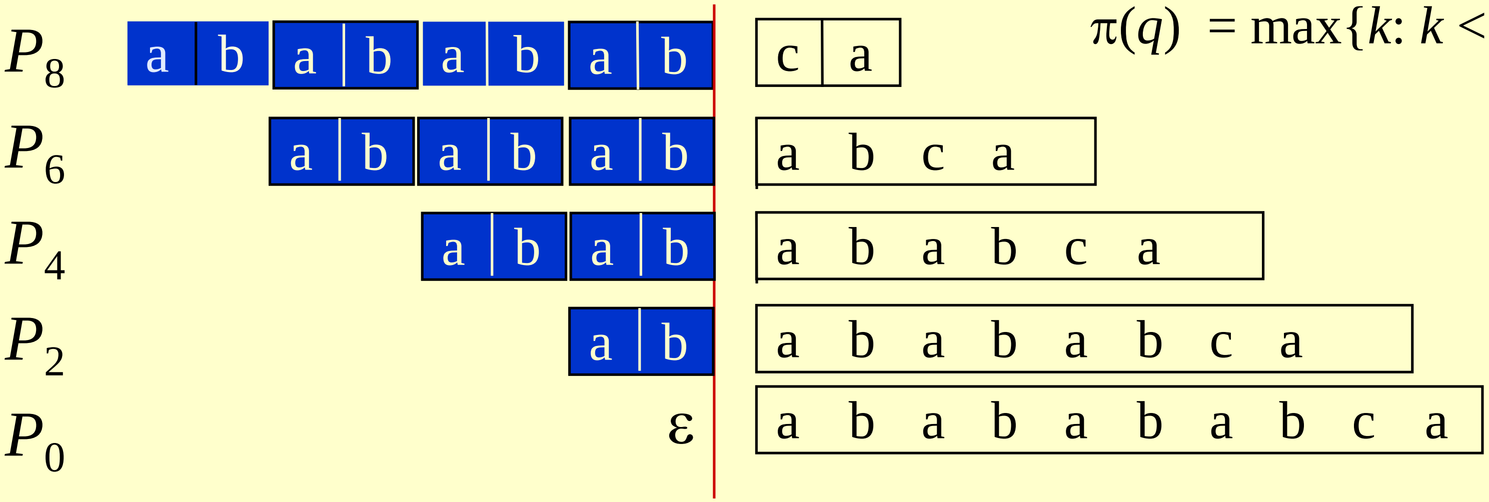
O(|Σ|(m+1))

1	1	3	1	5	1	7	1
0	2	0	4	0	4	0	2
0	0	0	0	0	6	0	0

i	1	2	3	4	5	6	7	8	9	10
$P[i]$	a	b	a	b	a	b	a	b	c	a
$\pi[i]$	0	0	1	2	3	4	5	6	0	1

O(m)

$\pi(q) = \max\{k: k < q, P_k \blacksquare P_q\}$



P_8 a b a b a b a b c a
P_6 a b a b a b a b c a
P_4 a b a b a b a b c a
P_2 a b a b a b a b c a
P_0 ε a b a b a b a b c a

$\pi[8] = 6$

$\pi[6] = 4$

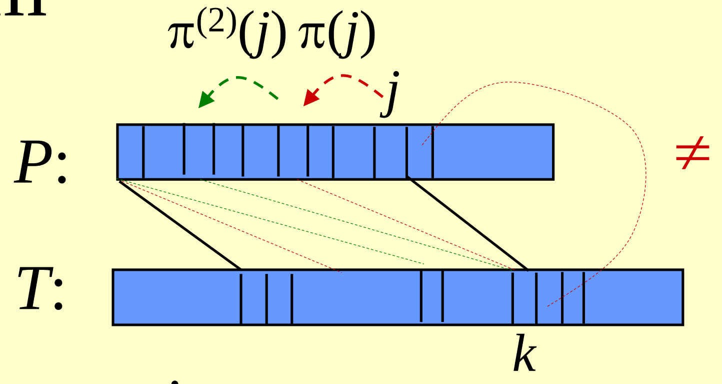
$\pi[4] = 2$

$\pi[2] = 0$

By using the values of prefix function values,
we will dynamically compute suffix function values.
In this way, a suffix function value is computed only
when it is needed. Thus, a lot of time can be saved.

How?

■ Knuth-Morris-Pratt algorithm



- function $\pi^{(u)}(j)$

  i) $\pi^{(1)}(j) = \pi(j)$, and

  ii) $\pi^{(u)}(j) = \pi(\pi^{(u-1)}(j))$, for $u > 1$.

  That is, $\pi^{(u)}(j)$ is just π applied u times to j.

  Example: $\pi^{(2)}(6) = \pi(\pi(6)) = \pi(4) = 2$ for P = ababababca.

- How to use $\pi^{(u)}(j)$?

  Suppose that the automaton is in state j, having read $T[1 .. k]$, and that $T[k+1] \neq P[j+1]$. Then, apply π repeatedly until it find the smallest value of u for which either

  1. $\pi^{(u)}(j) = l$ and $T[k + 1] = P[l + 1]$, or
  2. $\pi^{(u)}(j) = 0$ and $T[k + 1] \neq P[1]$.

■ Knuth-Morris-Pratt algorithm

- How to use $\pi^{(u)}(j)$?

1. $\pi^{(u)}(j) = l$ and $T[k+1] = P[l+1]$, or
2. $\pi^{(u)}(j) = 0$ and $T[k+1] \neq P[1]$.

That is, the automaton backs up through $\pi^{(1)}(j)$, $\pi^{(2)}(j)$, … until either Case 1 or 2 holds for $\pi^{(u)}(j)$ but not for $\pi^{(u-1)}(j)$.

- If Case 1 holds, the automaton enters state l.
- If Case 2 holds, it enters state 0.

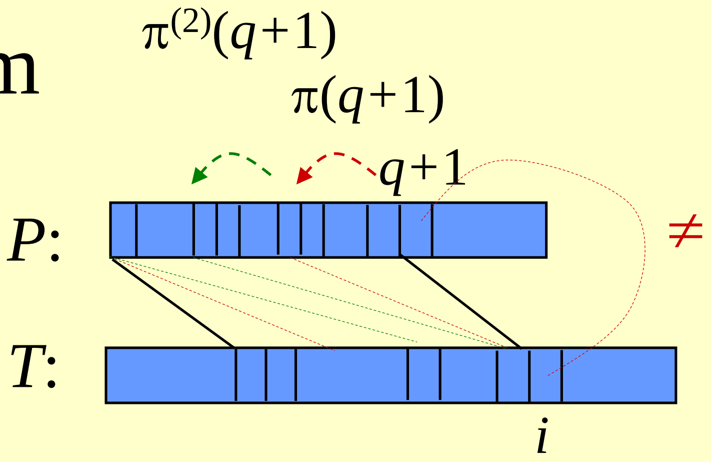
In either case, input pointer is advanced to position $T[k + 2]$.

In Case 1, $P[1 .. l]$ is the longest prefix of P that is a suffix of $T[1 .. k]$, then $P[1 .. \pi^{(u)}(j) + 1] = P[1 .. l + 1]$ is the longest prefix of P that is a suffix of $T[1 .. k + 1]$. In Case 2, no prefix of P is a suffix of $T[1 .. k + 1]$ and we will search P from scratch.

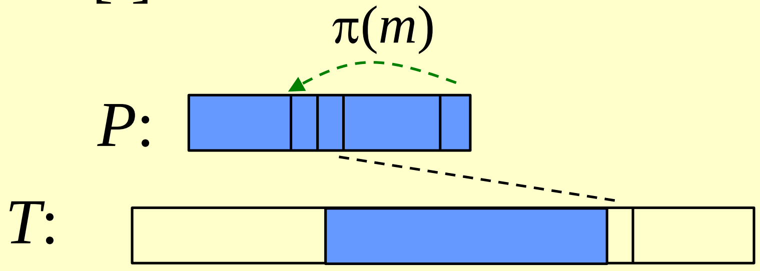
■ Knuth-Morris-Pratt algorithm

- Algorithm

```
KMP-Matcher($T$, $P$)
1.   $n \leftarrow length[T]$
2.   $m \leftarrow length[P]$
3.   $\pi \leftarrow$ Compute-Prefix-Function($P$)
4.   $q \leftarrow 0$
5.   for $i \leftarrow 1$ to $n$
6.       do while $q > 0$ and $P[q + 1] \neq T[i]$
7.               do $q \leftarrow \pi[q]$
8.           if $P[q + 1] = T[i]$
9.           then $q \leftarrow q + 1$
10.          if $q = m$
11.              then print "pattern occurs with shift" $i - m$
12.                  $q \leftarrow \pi[q]$
```



Compute $\pi^{(u)}(q+1)$



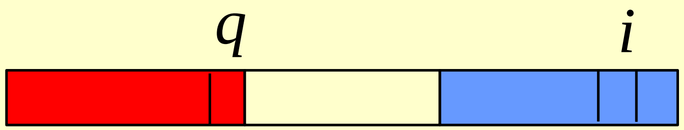
■ Knuth-Morris-Pratt algorithm

- Algorithm

```
4. $q \leftarrow 0$
5. **for** $i \leftarrow 1$ **to** $n$
6. **do while** $q > 0$ and $P[q + 1] \neq T[i]$
7.           **do** $q \leftarrow \pi[q]$
8.       **if** $P[q + 1] = T[i]$
9.       **then** $q \leftarrow q + 1$
10.      **if** $q = m$
11.      **then** print …
```

```
Compute-Prefix-Function($P$)
1.  $m \leftarrow length[T]$
2.  $\pi[1] \leftarrow 0$
3.  $q \leftarrow 0$
4.  **for** $i \leftarrow 2$ **to** $m$
5.      **do while** $q > 0$ and $P[q + 1] \neq P[i]$
6.              **do** $q \leftarrow \pi[q]$
7.          **if** $P[q + 1] = P[i]$
8.          **then** $q \leftarrow q + 1$
9.          $\pi[i] \leftarrow q$
10. **return** $\pi$
```

/*if $q = 0$ or $P[q + 1] = P[i]$, going out of the while-loop.*/



Compute $\pi^{(u)}(q+1)$

P = ababababca

i	1	2	3	4	5	6	7	8	9	10
$P[i]$	a	b	a	b	a	b	a	b	c	a
$\pi[i]$	0	0	1	2	3	4	5	6	0	1

■ Knuth-Morris-Pratt algorithm – sample trace

- Example

  P = ababababca,

  T = ababaababababca

Compute prefix function:

```
2. $\pi[1] = 0$
3. $q \leftarrow 0$
4. for $i \leftarrow 2$ to $n$
5. do while $q > 0$ and $P[q+1] \neq P[i]$
6.      do $q \leftarrow \pi[q]$
7.      if $P[q+1] = P[i]$
8.      then $q \leftarrow q + 1$
9.      $\pi[i] \leftarrow q$
```

$\pi[1] = 0$

$q = 0$

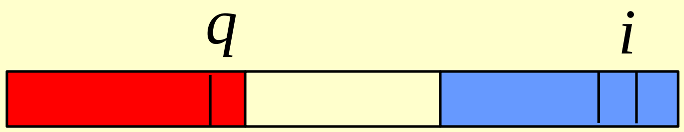
$i = 2$, $P[q + 1] = P[1] = \text{a}$, $P[i] = P[2] = \text{b}$, $P[q + 1] \neq P[i]$

$\pi[i] \leftarrow q$ ($\pi[2] \leftarrow 0$)

$i = 3$, $P[q + 1] = P[1] = \text{a}$, $P[i] = P[3] = \text{a}$, $P[q + 1] = P[i]$

$q \leftarrow q + 1$, $\pi[i] \leftarrow q$ ($\pi[3] \leftarrow 1$)

$q = 1$



$i = 4$, $P[q + 1] = P[2] = \text{b}$, $P[i] = P[4] = \text{b}$, $P[q + 1] = P[i]$

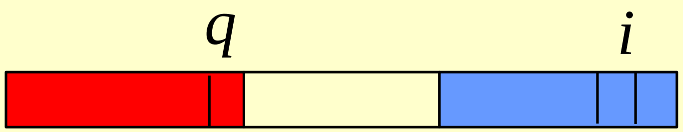
$q \leftarrow q + 1$, $\pi[i] \leftarrow q$ ($\pi[4] \leftarrow 2$)

P = ababababca

i	1	2	3	4	5	6	7	8	9	10
$P[i]$	a	b	a	b	a	b	a	b	c	a
$\pi[i]$	0	0	1	2	3	4	5	6	0	1

Knuth-Morris-Pratt algorithm – sample trace

- Example



$q = 2$

$i = 5$, $P[q + 1] = P[3] =$ a, $P[i] = P[5] =$ a, $P[q + 1] = P[i]$

$q \leftarrow q + 1$, $\pi[i] \leftarrow q$ ($\pi[5] \leftarrow 3$)

$q = 3$

$i = 6$, $P[q + 1] = P[4] =$ b, $P[i] = P[6] =$ b, $P[q + 1] = P[i]$

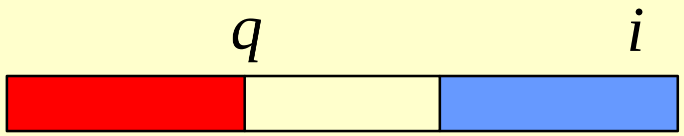
$q \leftarrow q + 1$, $\pi[i] \leftarrow q$ ($\pi[6] \leftarrow 4$)

$P =$ abababbabca

```
3. $q \leftarrow 0$
4. **for** $i \leftarrow 2$ **to** $n$
5. **do while** $q > 0$ and $P[q + 1] \neq P[i]$
6.        **do** $q \leftarrow \pi[q]$
**7.**        **if** $P[q + 1] = P[i]$
**8.**        **then** $q \leftarrow q + 1$
9.        $\pi[i] \leftarrow q$
```

Knuth-Morris-Pratt algorithm – sample trace

- Example



$q = 4$

$i = 7$, $P[q + 1] = P[5] = \text{a}$, $P[i] = P[7] = \text{a}$, $P[q + 1] = P[i]$

$q \leftarrow q + 1$, $\pi[i] \leftarrow q$ ($\pi[7] \leftarrow 5$)

$q = 5$

$i = 8$, $P[q + 1] = P[6] = \text{b}$, $P[i] = P[8] = \text{b}$, $P[q + 1] = P[i]$

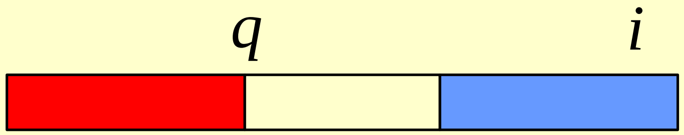
$q \leftarrow q + 1$, $\pi[i] \leftarrow q$ ($\pi[8] \leftarrow 6$)

```
3. $q \leftarrow 0$
4. **for** $i \leftarrow 2$ **to** $n$
5. **do while** $q > 0$ and $P[q + 1] \neq P[i]$
6.        **do** $q \leftarrow \pi[q]$
**7.**        **if** $P[q + 1] = P[i]$
**8.**        **then** $q \leftarrow q + 1$
9.        $\pi[i] \leftarrow q$
```

P = abababbabca

■ Knuth-Morris-Pratt algorithm – sample trace

- Example



$q = 6$

$i = 9$, $P[q + 1] = P[7] = \text{a}$, $P[i] = P[9] = \text{c}$, $P[q + 1] \neq P[i]$

$q \leftarrow \pi[q]$ ($q \leftarrow \pi[6] = 4$)

$P[q + 1] = P[5] = \text{a}$, $P[i] = P[9] = \text{c}$, $P[q + 1] \neq P[i]$

$q \leftarrow \pi[q]$ ($q \leftarrow \pi[4] = 2$)

$P[q + 1] = P[3] = \text{a}$, $P[i] = P[9] = \text{c}$, $P[q + 1] \neq P[i]$

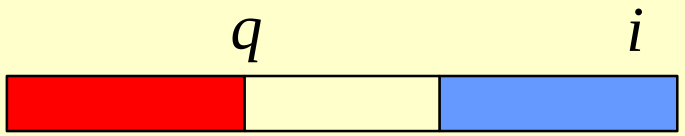
$q \leftarrow \pi[q]$ ($q \leftarrow \pi[2] = 0$), $\pi[i] \leftarrow q$ ($\pi[9] \leftarrow 0$)

```
3. $q \leftarrow 0$
4. **for** $i \leftarrow 2$ **to** $n$
5. **do while** $q > 0$ and $P[q + 1] \neq P[i]$
6.     **do** $q \leftarrow \pi[q]$
7.     **if** $P[q + 1] = P[i]$
8.     **then** $q \leftarrow q + 1$
9.     $\pi[i] \leftarrow q$
```

P = ababababca

Knuth-Morris-Pratt algorithm – sample trace

- Example



$q = 0$

$i = 10$, $P[q + 1] = P[1] = \mathrm{a}$, $P[i] = P[10] = \mathrm{a}$, $P[q + 1] = P[i]$

$q \leftarrow q + 1$, $\pi[i] \leftarrow q$ ($\pi[10] \leftarrow 1$)

$P = \mathrm{ababababca}$

```
3. $q \leftarrow 0$
4. **for** $i \leftarrow 2$ **to** $n$
5. **do while** $q > 0$ and $P[q + 1] \neq P[i]$
6.        **do** $q \leftarrow \pi[q]$
7.        **if** $P[q + 1] = P[i]$
8.        **then** $q \leftarrow q + 1$
9.        $\pi[i] \leftarrow q$
```

- Knuth-Morris-Pratt algorithm

  Theorem Algorithm Compute-Prefix-Function(P) computes π in O($|P|$) steps.

  Proof. The cost of the **while** statement is proportional to the number of times q is decremented by the statement $q \leftarrow \pi[q]$ following **do** in line 6. The only way k is increased is by assigning $q \leftarrow q + 1$ in line 8. Since $q = 0$ initially, and line 8 is executed at most ($|P| - 1$) times, we conclude that the **while** statement on lines 5 and 6 cannot be executed more than $|P|$ times. Thus, the total cost of executing lines 5 and 6 is O($|P|$). The remainder of the algorithm is clearly O($|P|$), and thus the whole algorithm takes O($|P|$) time.